Florentin Smarandache

# Neutrosophic Perspectives

## Triplets, Duplets, Multisets, Hybrid Operators, Modal Logic, Hedge Algebras. And Applications

*Second extended and improved edition*

pons

Florentin Smarandache

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Florentin Smarandache

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# TABLE OF CONTENTS

**PREFACE**
**Neutrosophic Perspectives** **15**

**CHAPTER I**
**Positively or Negatively Qualitative Neutrosophic Components** **19**
**The Average Positive Qualitative Neutrosophic Function and The Average Negative Qualitative Neutrosophic Function** **20**

**CHAPTER II**
**Neutrosophic Overnumbers / Undernumbers / Offnumbers** **25**
Single-Valued Neutrosophic Overnumbers / Undernumbers / Offnumbers 25
Interval-Valued Neutrosophic Overnumbers / Undernumbers / Offnumbers 26
Subset-Valued Neutrosophic Overnumbers / Undernumbers / Offnumbers 26
Spherical Neutrosophic Numbers 28
Single-Valued Spherical Neutrosophic Numbers 28
Interval-Valued Spherical Neutrosophic Numbers 28
Subset-Valued Spherical Neutrosophic Numbers 29

**CHAPTER III**
**Neutrosophic Indeterminacy of Second Type** **30**
Literal Indeterminacy (I) of first order 30
Numerical Indeterminacy of first order 30
Radical of Literal Indeterminacy 31
Literal Indeterminacies of second order 32

**CHAPTER IV**
**n-Refined Neutrosophic Set and Logic and Its Applications to Physics 33**
Two-Valued Logic 34
The Two Symbol-Valued Logic 34
The Two Numerical-Valued Logic 34
Three-Valued Logic 34
The Three Symbol-Valued Logics 34
The Three Numerical-Valued Logic 36
Four-Valued Logic 37
The Four Symbol-Valued Logic 37
Four Numerical-Valued Neutrosophic Logic 38
Five-Valued Logic 38
Seven-Valued Logic 39
n-Valued Logic 40
n-Valued Neutrosophic Logic Connectors 42
Particular Cases 46
Distinction between Neutrosophic Physics and Paradoxist Physics 46
Neutrosophic Physics 46
Paradoxist Physics 47
n-Valued Refined Neutrosophic Logic Applied to Physics 47

**CHAPTER V**
**Operations with Neutrosophic Numbers 52**
Addition of Neutrosophic Numbers 52
Subtraction of Neutrosophic Numbers 52
Multiplication of Neutrosophic Numbers 53
Division of Neutrosophic Numbers 53
Scalar Multiplication of Neutrosophic Numbers 53
Power of Neutrosophic Numbers 54
**Addition of Multiple Single-Valued Neutrosophic Numbers 55**
**Interval-Valued Neutrosophic Number Operations 57**
Addition of Intervals 57
Multiplication of Intervals 57
Subtraction of Intervals 57
Division of Intervals 58
Addition of Interval-Valued Neutrosophic Numbers 59

Multiplication of Interval-Valued Neutrosophic Numbers 59
Subtraction of Interval-Valued Neutrosophic Numbers: 59
Division of Interval-Valued Neutrosophic Numbers: 59
**Operations with (t, i, f)-Neutrosophic Matrices 60**
Addition of $(t,i,f)$-Neutrosophic Matrices 60
A More General Definition of Addition of $(t,i,f)$-Neutrosophic Matrices 60
Substraction of $(t,i,f)$-Neutrosophic Matrices 61
Scalar Multiplication of $(t,i,f)$-Neutrosophic Matrices 61
Multiplication of $(t,i,f)$-Neutrosophic Matrices 61
A More General Definition of Multiplication of $(t,i,f)$-Neutrosophic Matrices 61
Examples 63
Addition 63
Multiplication 63
Addition 65
Multiplication 65
Addition 67
Multiplication 67

**CHAPTER VI**
**Neutrosophic Hybrid Operators 69**
Neutrosophic Hybrid Intersection 70
Neutrosophic Hybrid Union 71
Neutrosophic Hybrid Implication 72
Neutrosophic Hybrid Equivalence 72

**CHAPTER VII**
**Neutrosophic Triplets 74**
Definition of Neutrosophic Triplet 74
Example of Neutrosophic Triplet 74
Definition of Neutrosophic Triplet Strong Set (or Neutrosophic Triplet Set) 75
Example of Neutrosophic Triplet Strong Set 75
Definition of Neutrosophic Triplet Weak Set 76
Theorem 76
Definition of Neutrosophic Triplet Strong Group (or Neutrosophic Triplet Group) 77
Example of Neutrosophic Triplet Strong Group 78

Proposition 79
Definition of Neutrosophic Perfect Triplet 79
Definition of Neutrosophic Imperfect Triplet 79
Examples of Neutrosophic Perfect and Imperfect Triplets 79
Definition of Neutrosophic Triplet Relationship of Equivalence 82
Example of Neutrosophic Triplet Relationship of Equivalence 83
Example of Neutrosophic Perfect and Imperfect Triplets 84
Example of Neutrosophic Perfect and Imperfect Triplets 84
Example of Non-Associative Law 85
Definition of Neutrosophic Enemy of Itself 87
Definition of Two Neutrosophic Friends 87
Definition of n≥2 Neutrosophic Friends 88
Proposition 89
Example of Neutrosophic Friends 89
**Neutrosophic Triplet Function 90**
**Theorems on Neutrosophic Triplets 91**
Definition 1 91
Definition 2 91
Definition 3 91
Definition 4 92
Theorem 1 92
Theorem 2 94
Counter-Example 1 96
Theorem 3 97
Definition of Neutro-Homomorphism. 98
Example 99
Definition 6 100
Proposition 1 100
Proposition 2 100
Theorem 4 102
Theorem 5 103
Theorem 6 104
**Neutrosophic Triplet Group vs. Generalized Group 105**
**Neutrosophic Triplet Multiple Order 109**

**CHAPTER VIII**
**Neutrosophic Triplet Ring** **111**
Definition of Neutrosophic Triplet Ring 111
**Hybrid Neutrosophic Triplet Ring** **113**
Definition 113
**Hybrid Neutrosophic Triplet Ring of Second Type** **115**
Definition 115
**Neutrosophic Triplet Field** **117**
Definition 117
Example of Neutrosophic Triplet Ring which is not a Neutrosophic Triplet Field. 119
**Hybrid Neutrosophic Triplet Field** **121**
Hybrid Neutrosophic Triplet Field of Type 1. 121
Hybrid Neutrosophic Triplet Field of Type 2. 121
**Neutrosophic Triplet Loop** **122**
**Neutrosophic Triplet Structures** **124**

**CHAPTER IX**
**Neutrosophic Duplets** **127**
Definition of Neutrosophic Duplet. 127
Example of Neutrosophic Duplets. 127
**Neutrosophic Duplet Set and Neutrosophic Duplet Structures** **130**
Definition of Neutrosophic Duplet Strong Set 130
Definition of Neutrosophic Duplet Weak Set 130
Proposition. 131
Theorem 131
Example of Neutrosophic Duplet Strong Set 132

**CHAPTER X**
**Neutrosophic Multiset** **135**
Definition 135
Examples 135
Examples of operations with neutrosophic multisets. 137
Intersection of Neutrosophic Multisets. 137
Union of Neutrosophic Multisets. 137
Inclusion of Neutrosophic Multisets. 137

Cardinality of Neutrosophic Multisets. 138
Cartesian Product of Neutrosophic Multisets. 138
Difference of Neutrosophic Multisets. 138
Sum of Neutrosophic Multisets. 138
**Neutrosophic Multiset Applied in Physical Processes 140**
**Neutrosophic Complex Multiset 142**
Example of Neutrosophic Complex Set. 142
Examples of Neutrosophic Complex Multiset. 142

**CHAPTER XI**
**Classical Logic and Neutrosophic Logic. Answers to K. Georgiev. 144**
Single Valued Neutrosophic Set. 144
Standard and Non-Standard Real Subsets. 147
Degrees of Membership > 1 or < 0 of the Elements. 149
Neutrosophic Logic Negations. 150
Degree of Dependence and Independence between (Sub)Components. 152
Definition of Neutrosophic Logic. 154
Neutrosophic Logical Systems. 155
Conclusion. 161
**Neutrosophic Tautological, Contradictional, and Provability Tresholds, and Neutrosophic Completeness. Answers to U. Rivieccio 165**
Belnap's logic system 165
Kleene's three-valued logic, 166
Paraconsistent Logics 166
Incomplete Logics 167
Set-Valued Neutrosophic Logic 167
Non-standard Set-Valued Neutrosophic Logic 168
Neutrosophic Negation Operator 170
Neutrosophic Conjunction Operator 171
Neutrosophic Disjunction Operator 173
Incomplete Neutrosophic Logic 174
Neutrosophic Implication 175
Neutrosophic "Tautology" 176
Neutrosophic Propositional Logic 177
Neutrosophic "Completeness" 179

Laws of Classical Logic That Do Not Hold in The Interval Neutrosophic Logic 184
Modal Contexts 186
Neutrosophic Score Function 186
Applications. 187
Neutrosophic Lattices 188
Conclusion 191

**CHAPTER XII**
**Neutrosophic Predicate Logic 196**
Neutrosophic Quantifiers 199
Neutrosophic Existential Quantifier. 199
Neutrosophic Universal Quantifier. 199
**Neutrosophic Decidability System 200**
**Neutrosophic Modal Logic 202**
Introduction. 202
Neutrosophic Alethic Modal Operators 205
Neutrosophic Possibility Operator. 205
Neutrosophic Necessity Operator 207
Other Possibility and Necessity Operators 208
Connection between Neutrosophic Possibility Operator and Neutrosophic Necessity Operator. 210
Neutrosophic Modal Equivalences 213
Neutrosophic Truth Threshold 214
Neutrosophic Semantics. 215
Formalization 216
Neutrosophic Formulas. 216
Accesibility Relation in a Neutrosophic Theory. 217
Applications. 218
Neutrosophic n-ary Accesibility Relation. 218
Neutrosophic Kripke Frame. 219
Neutrosophic (t, i, f)-Assignement. 219
Neutrosophic Deducibility. 220
Refined Neutrosophic Modal Single-Valued Logic 222
Refined Neutrosophic Possibility Operator. 222
Refined Neutrosophic Necessity Operator. 223
Application of the Neutrosophic Threshold. 224
Neutrosophic Mereology 225
Neutrosophic Mereological Threshold 225
Neutrosophic Gnosisology 226
Neutrosophic Gnosisological Threshold 227

Neutrosophic Epistemology 227
Neutrosophic Epistemological Threshold 228
Conclusions. 228
**Neutrosophic Hedge Algebras 233**
Introduction 233
Materials and Methods 233
Operations on Neutrosophic Components 238
Neutrosophic Hedge-Element Operators 238
Neutrosophic Increment 238
Neutrosophic Decrement 239
Theorem 1 239
Neutrosophic Hedge-Hedge Operators 239
Neutrosophic Hedge Operators 239
Theorem 2 240
Theorem 3 241
Converse Hedges 241
Compatible Hedges 241
Commutative Hedges 241
Cumulative Hedges 241
Neutrosophic Hedge Algebra 242
Example of a Neutrosophic Hedge Algebra $\tau$ 242
Theorem 4 246
Theorem 5 246
Diagram of the Neutrosophic Hedge Algebra $\tau$ 247
Conclusions 248

**CHAPTER XIII: APPLICATIONS**
**Neutrosophic MCDM 249**
**Neutrosophic Psychology 250**
**Neutrosophic Function as The Equatorial Virtual Line 251**
**PCR5 and Neutrosophic Probability in Target Identification 252**
Introduction 252
Definition of Neutrosophic Measure 254
Normalized Neutrosophic Measure 256
Normalized Probability 257
The PCR5 Formula 258
Application in Information Fusion 259
Example 261

Conclusion 264
**Easier to Break a Dynamic System from Inside than from Outside 266**
Modeling Methodology. 269
Model of Breaking a Neutrosophic Complex Dynamic System 271
Ordinary Differential Equations Model 281
Operations with Single-Valued Neutrosophic Probabilities 282
Operations with Subset-Valued Neutrosophic Probabilities 283
Whole Neutrosophic Hyperrelationships 284
Theorem 286
Breaking Point Equilibrium Threshold 287
Examples of Complex Dynamic System 292
Methods Used for Breaking from Inside a Complex Dynamic System 292
Extension of the Model 297
Equilibrium Points 298
Comments on the Model 298
Conclusion 299
**Neutrosophic Quantum Computer 302**
Introduction. 302
Neutrosophic Polarization. 303
Refined Neutrosophic Polarization. 303
Neutrosophic Quantum Computer. 304
Neutrosophic Particle. 305
Entangled Neutrosophic Particle. 305
Neutrosophic Data. 305
Neutrosophic Superposition. 305
Indeterminate Bit. 306
Neutrobit. 306
Refined Neutrosophic Quantum Computer. 307
Neutrosophic Filter Polarization. 307
Neutrosophic Quantum Parallelism. 307
n-Neutrobit Quantum Computer. 308
Neutrosophic Quantum Gates. 308
Remarks. 308
Reversibility of a Neutrosophic Quantum Computer. 309
Neutrosophic Dynamical System. 310

Neutrosophic Turing Machine & Neutrosophic Church-Turing Principle. 310
Human Brain as an example of Neutrosophic Quantum Computer. 311
Neutrosophic Quantum Dot. 311
Neutrosophic NOT Function. 313
Neutrosophic AND Function. 313
Neutrosophic OR Function. 314
Neutrosophic IFTHEN Function. 314
Neutrosophic Quantum Liquids. 316
Conclusion. 317
**Theory of Neutrosophic Evolution: Degrees of Evolution, Indeterminacy, and Involution 318**
Introduction. 319
Taxonomy. 322
Species. 322
Theories of Origin of Life. 323
Theories and Ideas about Evolution. 324
Open Research. 332
Involution. 332
Theory of Neutrosophic Evolution. 333
Dynamicity of the Species. 335
Several Examples of Evolution, Involution, and Indeterminacy (Neutrality) 336
Cormorants Example 336
Cosmos Example. 337
Example of Evolution and Involution 337
Penguin Example. 338
Frigate Birds Example. 338
Example of Darwin's Finches. 338
El Niño Example. 339
Bat Example. 340
Mole Example. 340
Neutrosophic Selection 341
Refined Neutrosophic Theory of Evolution 341
Open Questions on Evolution / Neutrality / Involution. 342
Conclusion. 343
**Neutrosophic Triplet Structures in Practice 346**

**PREFACE**

## *Neutrosophic Perspectives*

This book is part of the book-series dedicated to the advances of neutrosophic theories and their applications, started by the author in 1998.

Its aim is to present the last developments in the field. For the first time, we now introduce:

— Neutrosophic Duplets and the Neutrosophic Duplet Structures;

— Neutrosophic Multisets (as an extension of the classical multisets);

— Neutrosophic Spherical Numbers;

— Neutrosophic Overnumbers / Undernumbers / Offnumbers;

— Neutrosophic Indeterminacy of Second Type;

— Neutrosophic Hybrid Operators (where the heterogeneous t-norms and t-conorms may be used in designing neutrosophic aggregations);

— Neutrosophic Triplet Weak Set (and consequently we have renamed the previous Neutrosophic Triplet Set (2014-2016) as Neutrosophic Triplet Strong Set in order to distinguish them);

— Neutrosophic Perfect Triplet;
— Neutrosophic Imperfect Triplet;
— Neutrosophic triplet relation of equivalence;
— Two Neutrosophic Friends;
— n Neutrosophic Friends;
— Neutrosophic Triplet Loop;
— Neutrosophic Triplet Function;
— Neutrosophic Modal Logic;
— and Neutrosophic Hedge Algebras.

The Refined Neutrosophic Set / Logic / Probability were introduced in 2013 by F. Smarandache. Since year 2016 a new interest has been manifested by researchers for the Neutrosophic Triplets and their corresponding Neutrosophic Triplet Algebraic Structures (introduced by F. Smarandache & M. Ali). Subtraction and Division of Neutrosophic Numbers were introduced by F. Smarandache - 2016, and Jun Ye – 2017. We also present various new applications (except the first one) in: neutrosophic multi-criteria decision-making, neutrosophic psychol-ogy, neutrosophic geographical function (the equatorial virtual line), neutrosophic probability in target identification,

neutrosophic dynamic systems, neutrosophic quantum computers, neut-rosophic theory of evolution, and neutrosophic triplet structures in our everyday life. In this version, we make a distinction between 'neutrosophic triplet strong set' together with the algebraic structures defined on it, and 'neutrosophic triplet weak set' together with the algebraic structures defined on it.

*The Author*

**CHAPTER I**

# *I.1. Positively or Negatively Qualitative Neutrosophic Components*

Here it is the general picture on the neutrosophic components *T, I, F* :

- the *T* is considered a positively (good) qualitative component;

- while *I* and *F* are considered the opposite, i.e. negatively (bad) qualitative components.

When we apply neutrosophic operators, for *T* we apply one type, while for *I* and *F* we apply an opposite type.

Let's see examples:

*- neutrosophic conjunction:*

$\langle t_1, i_1, f_1 \rangle \wedge \langle t_2, i_2, f_2 \rangle =< t_1 \wedge t_2, i_1 \vee i_2, f_1 \vee f_2 >$; (1)

as you reader see we have *t*-norm for $t_1$ and $t_2$, but *t*-conorm for $i_1$ and $i_2$, as well as for $f_1$ and $f_2$;

*- neutrosophic disjunction:*

$\langle t_1, i_1, f_1 \rangle \vee \langle t_2, i_2, f_2 \rangle =< t_1 \vee t_2, i_1 \wedge i_2, f_1 \wedge f_2 >$; (2)

Etc.

# *I.2. The Average Positive Qualitative Neutrosophic Function and The Average Negative Qualitative Neutrosophic Function*

The *Average Positive Quality Neutrosophic Function* (also known as *Neutrosophic Score Function*, which means expected/average value) of a neutrosophic number.

Let *(t, i, f)* be a single-valued neutrosophic number, where *t, i, f ∈ [0, 1].*

The component *t* (truth) is considered as a positive quality, while *i* (indeterminacy) and *f* (falsehood) are considered negative qualities.

Contrarily, *1-t* is considered a negative quality, while *1-i* and *1-f* are considered positive qualities.

Then, the average positive quality function of a neutrosophic number is defined as: (1)

$$s^+ : [0,1]^3 \to [0,1], s^+(t,i,f) = \frac{t+(1-i)+(1-f)}{3} = \frac{2+t-i-f}{3}.$$

We now introduce for the first time the *Average Negative Quality Neutrosophic Function* of a neutrosophic number**,** defined as: (2)

$$s^{-}:[0,1]^3 \to [0,1], s^{-}(t,i,f) = \frac{(1-t)+i+f}{3} = \frac{1-t+i+f}{3}.$$

*Theorem I.2.1.*

The average positive quality neutrosophic function and the average negative quality neutrosophic function are complementary to each other, or

$$s^{+}(t,i,f) + s^{-}(t,i,f) = 1. \quad (3)$$

*Proof.*

$$s^{+}(t,i,f) + s^{-}(t,i,f) = \frac{2+t-i-f}{3} + \frac{1-t+i+f}{3} = 1. \quad (4)$$

The *Neutrosophic Accuracy Function* has been defined by:

$h: [0, 1]^3 \to [-1, 1], h(t, i, f) = t - f.$ (5)

We introduce now for the first time the *Extended Accuracy Neutrosophic Function*, defined as follows:

$h_e: [0, 1]^3 \to [-2, 1], h_e(t, i, f) = t - i - f,$ (6)

which varies on a range: from the worst negative quality (-2) [or minimum value], to the best positive quality [or maximum value].

The *Neutrosophic Certainty Function* is:

$c: [0, 1]^3 \rightarrow [0, 1]$, $c(t, i, f) = t$. (7)

*Generalization.*

The above functions can be extended for the case when the neutrosophic components *t, i, f* are intervals (or, even more general, subsets) of the unit interval *[0, 1].*

*Total Order.*

Using three functions from above: neutrosophic score function, neutrosophic accuracy function, and neutrosophic certainty function, one can define a total order on the set of neutrosophic numbers.

In the following way:

Let $(t_1, i_1, f_1)$ and $(t_2, i_2, f_2)$, where $t_1, i_1, f_1, t_2, i_2, f_2 \in [0, 1]$, be two single-valued neutrosophic numbers. Then:

- 1. If $s^+(t_1, i_1, f_1) > s^+(t_2, i_2, f_2)$, then $(t_1, i_1, f_1) >_N (t_2, i_2, f_2)$;
- 2. If $s^+(t_1, i_1, f_1) = s^+(t_2, i_2, f_2)$ and $h(t_1, i_1, f_1) > h(t_2, i_2, f_2)$, then $(t_1, i_1, f_1) >_N (t_2, i_2, f_2)$;
- 3. If $s^+(t_1, i_1, f_1) = s^+(t_2, i_2, f_2)$ and $h(t_1, i_1, f_1) = h(t_2, i_2, f_2)$ and $c(t_1, i_1, f_1) > c(t_2, i_2, f_2)$, then $(t_1, i_1, f_1) >_N (t_2, i_2, f_2)$;

– 4. If $s^+(t_1, i_1, f_1) = s^+(t_2, i_2, f_2)$ and $h(t_1, i_1, f_1) = h(t_2, i_2, f_2)$ and $c(t_1, i_1, f_1) = c(t_2, i_2, f_2)$, then $(t_1, i_1, f_1) = (t_2, i_2, f_2)$.

*Applications.*

All the above functions are used in the ranking (comparison) of two neutrosophic numbers in multi-criteria decision making.

*Example of Comparison of Single-Valued Neutrosophic Numbers.*

Let's consider two single-valued neutrosophic numbers: <0.6, 0.1, 0.4> and <0.5, 0.1, 0.3>.

The neutrosophic score functions is:

$s^+(0.6, 0.1, 0.4) = (2 + 0.6 - 0.1 - 0.4) / 3 =$

$= 2.1 / 3 = 0.7;$

$s^+(0.5, 0.1, 0.3) = (2 + 0.5 - 0.1 - 0.3) / 3 =$

$= 2.1 / 3 = 0.7;$

Since $s^+(0.6, 0.1, 0.4) = s^+(0.5, 0.1, 0.3)$, we need to compute the neutrosophic accuracy functions:

$a(0.6, 0.1, 0.4) = 0.6 - 0.4 = 0.2;$

$a(0.5, 0.1, 0.3) = 0.5 - 0.3 = 0.2.$

Since $a(0.6, 0.1, 0.4) = a(0.5, 0.1, 0.3)$, we need to compute the neutrosophic certainty functions:

$c(0.6, 0.1, 0.4) = 0.6;$

c(0.5, 0.1, 0.3) = 0.5.

Because c(0.6, 0.1, 0.4) > c(0.5, 0.1, 0.3), we eventually conclude that the first neutrosophic number is greater than the second neutrosophic number, or:

$(0.6, 0.1, 0.4) >_N (0.5, 0.1, 0.3)$.

So, we need three functions in order to make a total order on the set of neutrosophic numbers.

## *References*

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2. Jiqian Chen and Jun Ye, *Some Single-Valued Neutrosophic Dombi Weighted Aggregation Operators for Multiple Attribute Decision-Making*, Licensee MDPI, Basel, Switzerland, Symmetry, 9, 82, 2017; DOI:10.3390/sym9060082

**CHAPTER II**

# *II.1. Neutrosophic Overnumbers / Undernumbers / Offnumbers*

## *II.1.1. Single-Valued Neutrosophic Overnumbers / Undernumbers / Offnumbers*

In 2007, we have introduced the *Neutrosophic Over/Under/Off-Set* and *Logic* [1, 2] that were totally different from other sets/logics.

The Neutrosophic Set was extended to *Neutrosophic Overset* (when some neutrosophic component is > 1), and to *Neutrosophic Underset* (when some neutrosophic component is < 0), and to *Neutrosophic Offset* (when some neutrosophic components are off the interval [0, 1], i.e. some neutrosophic component > 1 and some neutrosophic component < 0).

All such single-valued neutrosophic triplets *(t, i, f),* where *t, i, f* are single-value real numbers, with some *t, i,* or *f > 1* were called single-valued neutrosophic overnumbers, while with some *t, i,*

or $f < 0$ were called single-valued neutrosophic undernumbers, and with some $t, i, f > 1$ and other $< 0$ were called single-valued neutrosophic offnumbers.

### *II.1.2. Interval-Valued Neutrosophic Overnumbers / Undernumbers / Offnumbers*

The interval-valued neutrosophic triplets $(T, I, F)$, where $T, I, F$ are real intervals, with some $T, I,$ or $F$ intervals getting over $1$, were called interval-valued neutrosophic overnumbers, while with some $T, I,$ or $F$ intervals getting under $0$, were called interval-valued neutrosophic undernumbers, and with some $T, I, F$ intervals getting over $1$ while others getting under $0$, were called interval-valued neutrosophic offnumbers.

### *II.1.3. Subset-Valued Neutrosophic Overnumbers / Undernumbers / Offnumbers*

The subset-valued neutrosophic triplets $(T, I, F)$, where $T, I, F$ are real subsets {not necessarily

intervals}, with some *T, I,* or *F* subsets getting over *1*, were called subset-valued neutrosophic overnumbers, while with some *T, I,* or *F* subsets getting under *0*, were called subset-valued neutrosophic undernumbers, and with some *T, I, F* subsets getting over *1* while others getting under *0*, were called subsets-valued neutrosophic offnumbers.

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1. F. Smarandache, *Neutrosophy. Neutrosophic Probability, Set, and Logic*, ProQuest Information & Learning, Ann Arbor, Michigan, USA, 156 p., 2007; *http://fs.gallup.unm.edu/eBook-neutrosophics6.pdf* (sixth edition online);

2. F. Smarandache, *Neutrosophic Overset, Neutrosophic Underset, and Neutrosophic Offset. Similarly for Neutrosophic Over-/Under-/Off- Logic, Probability, and Statistics*, 168 p., Pons Editions, Bruxelles, Belgique, 2016; *https://hal.archives-ouvertes.fr/hal-01340830 ; https://arxiv.org/ftp/arxiv/papers/1607/1607.00234.pdf*

# *II.2. Spherical Neutrosophic Numbers*

## *II.2.1. Single-Valued Spherical Neutrosophic Numbers*

As a particular case of single-valued neutrosophic overnumbers, we present now for the first time the single-valued spherical neutrosophic numbers, which have the form *(t, i, f)*:

where the real single-values

$$t, i, f \in [0, \sqrt{3}],$$

and (1)

$$t^2 + i^2 + f^2 \leq 3.$$

They are generalization of Single-Valued Pythagorean Fuzzy Numbers *(t, f)*:

with $t, f \in [0, \sqrt{2}]$

and $t^2 + f^2 \leq 2.$ (2)

## *II.2.2. Interval-Valued Spherical Neutrosophic Numbers*

As a particular case of interval-valued neutrosophic overnumbers, we present now for the first

time the interval-valued spherical neutrosophic numbers, which have the form *(T, I, F):*

where the real intervals

$$T, I, F \subseteq [0, \sqrt{3}],$$

and (3)

$$T^2 + I^2 + F^2 \subseteq [0, 3].$$

## *II.2.3. Subset-Valued Spherical Neutrosophic Numbers*

As a particular case of subset-valued neutrosophic overnumbers, we present now for the first time the subset-valued spherical neutrosophic numbers, which have the form *(T, I, F):*

where the real subsets $T, I, F \subseteq [0, \sqrt{3}]$,

and (4)

$$T^2 + I^2 + F^2 \subseteq [0, 3].$$

**CHAPTER III**

# *III.1. Neutrosophic Indeterminacy of Second Type*

There are two types of neutrosophic indeterminacies:

## *III.1.1. Literal Indeterminacy (I) of first order*

*Example*: $2 + 3I$, where $I^2 = I$, and $I$ is a letter that does not represent a number.

## *III.1.2. Numerical Indeterminacy of first order*

*Example*: the element $x(0.6,0.3,0.4) \in A$,

meaning that $x$'s indeterminate-membership = 0.3.

Or the functions f(.) defined as: $f(6) = 7$ or $9$, or $f(0\ or\ 1) = 5$, or $f(x) = [0.2, 0.3]x^2$ etc.

Let's compute some neutrosophic limits (with numerical indeterminacies):

$$\lim_{x\to 0^+} \frac{[2.1, 2.5]}{\frac{1}{\ln x}} = \frac{[2.1, 2.5]}{\frac{1}{\ln 0}} = \frac{[2.1, 2.5]}{\frac{1}{-\infty}} = \frac{[2.1, 2.5]}{-0}$$

$$= \left[\frac{2.1}{-0}, \frac{2.5}{-0}\right] = (-\infty, -\infty) = -\infty.$$

Herein $[2.1, 2.5]$ is a numerical indeterminacy, not a literal indeterminacy.

$$\lim_{x\to[9,11]}[3.5,5.9]\,x^{[1,2]} = [3.5,5.9]\cdot[9,11]^{[1,2]} =$$

$$[3.5,5.9]\cdot[9^1,11^2] = [3.5,5.9]\cdot[9,121] =$$

$$[3.5(9),5.9(121)] = [31.5,713.9].$$

$$\lim_{x\to\infty}[3.5,5.9]x^{[1,2]} = [3.5,5.9]\cdot\infty^{[1,2]} = [3.5,5.9]\cdot\infty$$

$$= [3.5(\infty),5.9(\infty)] = [\infty,\infty] = \infty.$$

### *III.1.3. Radical of Literal Indeterminacy*

$$\sqrt{I} = x + yI$$

We need to find $x$ and $y$ by coefficient-identification method. After raising to the second power both sides we get:

$0 + 1\cdot I = x^2 + (2xy + y^2)I$

$x = 0, y = \pm 1,$

so $\sqrt{I} = \pm I.$

$\sqrt[3]{I}$=x+yI

We raise to the cube both sides:

$0 + 1\cdot I = x^3 + 3x^2y + 3xy^2I^2 + y^3I^3 = x^3 + (3x^2y + 3xy^2 + y^3)I$

Then we get:

$x = 0,\ y = 1,$

so $\sqrt[3]{I} = I.$

In general: $\sqrt[2k]{I} = \pm I$ and

$\sqrt[2k+1]{I} = I.$

### *III.1.4. Literal Indeterminacies of second order*

$$I^0, I^{-n} \text{ for } n > 0, 0^I, \frac{I}{0}, I \cdot \infty, \frac{I}{\infty}, \frac{\infty}{I}, \infty^I, I^\infty,$$

$$I^I, a^I (a \in \mathbb{R}), \infty \pm a \cdot I$$

are literal indeterminacies of second order.

**CHAPTER IV**

# *IV.1. n-Refined Neutrosophic Set and Logic and Its Applications to Physics*

## *Abstract*

In this paper, we present a short history of logics: from particular cases of 2-symbol or numerical valued logic to the general case of *n*-symbol or numerical valued logic.

We show generalizations of 2-valued Boolean logic to fuzzy logic, also from the Kleene's and Lukasiewicz' 3-symbol valued logics or Belnap's 4-symbol valued logic to the most general *n*-symbol or numerical valued refined neutrosophic logic.

Two classes of neutrosophic norm (*n*-norm) and neutrosophic conorm (*n*-conorm) are defined.

Examples of applications of neutrosophic logic to physics are listed in the last section.

Similar generalizations can be done for *n*-Valued Refined Neutrosophic Set, and respectively *n*-Valued Refined Neutrosophic Probability.

## *IV.1.1. Two-Valued Logic*

### *IV.1.1.1. The Two Symbol-Valued Logic*

It is the Chinese philosophy: *Yin and Yang* (or Femininity and Masculinity) as contraries:

*Fig. 1: Ying and Yang*

It is also the Classical or *Boolean Logic*, which has two symbol-values: truth *T* and falsity *F*.

### *IV.1.1.2. The Two Numerical-Valued Logic*

It is also the Classical or *Boolean Logic*, which has two numerical-values: truth 1 and falsity 0. More general it is the *Fuzzy Logic*, where the truth (*T*) and the falsity (*F*) can be any numbers in [0,1] such that $T + F = 1$.

Even more general, *T* and *F* can be subsets of [0,1].

## *IV.1.2. Three-Valued Logic*

### *IV.1.2.11 The Three Symbol-Valued Logics*

1. *Lukasiewicz's Logic*: True, False, and Possible.

2. *Kleene's Logic*: True, False, Unknown (or Undefined).

3. Chinese philosophy extended to: *Yin*, *Yang*, and *Neuter* (or Femininity, Masculinity, and Neutrality) - as in Neutrosophy.

Neutrosophy philosophy was born from neutrality between various philosophies. Connected with *Extenics* (Prof. Cai Wen, 1983), and *Paradoxism* (F. Smarandache, 1980). *Neutrosophy* is a new branch of philosophy that studies the origin, nature, and scope of neutralities, as well as their interactions with different ideational spectra. This theory considers every notion or idea <A> together with its opposite or negation <antiA> and with their spectrum of neutralities <neutA> in between them (i.e. notions or ideas supporting neither <A> nor <antiA>). The <neutA> and <antiA> ideas together are referred to as nonA. Neutrosophy is a generalization of Hegel's dialectics (the last one is based on <A> and <antiA> only). According to neutrosophy every idea <A> tends to be neutralized and balanced by <antiA> and <neutA> ideas - as a state of

equilibrium. In a classical way <A>, <neutA>, <antiA> are disjoint two by two. But, since in many cases the borders between notions are vague, imprecise, Sorites, it is possible that <A>, <neutA>, <antiA> (and <nonA> of course, where <nonA> = <neutA> ∪ <antiA>) have common parts two by two, or even all three of them as well. *Such contradictions involve Extenics.* Neutrosophy is the base of all neutrosophics and it is used in engineering applications (especially for software and information fusion), medicine, military, airspace, cybernetics, physics.

### *IV.1.2.2. The Three Numerical-Valued Logic*

1. *Kleene's Logic*: True (1), False (0), Unknown (or Undefined) (1/2), and uses "min" for ∧, "max" for ∨, and "1-" for negation.

2. More general is the *Neutrosophic Logic* [Smarandache, 1995], where the truth (*T*) and the falsity (F) and the indeterminacy (*I*) can be any numbers in [0, 1], then $0 \leq T + I + F \leq 3$. More general: Truth (*T*), Falsity (*F*), and Indeterminacy (*I*) are standard or nonstandard subsets of the nonstandard interval $]^{-}0, 1^{+}[$.

## *IV.1.3. Four-Valued Logic*

### *IV.1.3.1. The Four Symbol-Valued Logic*

1. It is *Belnap's Logic*: True (*T*), False (*F*), Unknown (*U*), and Contradiction (*C*), where *T, F, U, C* are symbols, not numbers. Below is the Belnap's conjunction operator table:

| ∩ | F | U | C | T |
|---|---|---|---|---|
| F | F | F | F | F |
| U | F | U | F | U |
| C | F | F | C | C |
| T | F | U | C | T |

Restricted to *T, F, U*, and to *T, F, C*, the Belnap connectives coincide with the connectives in Kleene's logic.

2. Let *G* = Ignorance. We can also propose the following two 4-Symbol Valued Logics: (*T, F, U, G*), and (*T, F, C, G*).

3. *Absolute-Relative 2-, 3-, 4-, 5-, or 6-Symbol Valued Logics* [Smarandache, 1995]. Let $T_A$ be truth in all possible worlds (according to Leibniz's definition); $T_R$ be truth in at least one world but not in all worlds; and similarly let $I_A$ be indeterminacy in all possible worlds; $I_R$ be

indeterminacy in at least one world but not in all worlds; also let $F_A$ be falsity in all possible worlds; $F_R$ be falsity in at least one world but not in all worlds. Then we can form several Absolute-Relative 2-, 3-, 4-, 5-, or 6-Symbol Valued Logics just taking combinations of the symbols $T_A$, $T_R$, $I_A$, $I_R$, $F_A$, and $F_R$. As particular cases, very interesting would be to study the Absolute- Relative 4-Symbol Valued Logic ($T_A$, $T_R$, $F_A$, $F_R$), as well as the Absolute-Relative 6-Symbol Valued Logic ($T_A$, $T_R$, $I_A$, $I_R$, $F_A$, $F_R$).

### *IV.1.3.2. Four Numerical-Valued Neutrosophic Logic*

Indeterminacy $I$ is refined (split) as $U$ = Unknown, and $C$ = contradiction. $T$, $F$, $U$, $C$ are subsets of [0, 1], instead of symbols. This logic generalizes Belnap's logic since one gets a degree of truth, a degree of falsity, a degree of unknown, and a degree of contradiction. Since $C = T \wedge F$, this logic involves the Extenics.

## *IV.1.4. Five-Valued Logic*

1. Five Symbol-Valued Neutrosophic Logic [Smarandache, 1995]: Indeterminacy $I$ is refined

(split) as $U$ = Unknown, $C$ = contradiction, and $G$ = ignorance; where the symbols represent:

$T$ = truth;

$F$ = falsity;

$U$ = neither $T$ nor $F$ (undefined);

$C = T \wedge F$, which involves the Extenics;

$G = T \vee F$.

2. If $T, F, U, C, G$ are subsets of [0, 1] then we get a *Five Numerical-Valued Neutrosophic Logic*.

## *IV.1.5. Seven-Valued Logic*

*A. Seven Symbol-Valued Neutrosophic Logic*

[Smarandache, 1995]:

$I$ is refined (split) as $U, C, G$, but $T$ also is refined as $T_A$ = absolute truth and $T_R$= relative truth, and $F$ is refined as $F_A$= absolute falsity and $F_R$= relative falsity. Where: $U$ = neither ($T_A$ or $T_R$) nor ($F_A$ or $F_R$) (i.e. undefined); C = ($T_A$ or $T_R$) $\wedge$ ($F_A$ or $F_R$) (i.e. Contradiction), which involves the Extenics; G = ($T_A$ or $T_R$)$\vee$($F_A$ or $F_R$) (i.e. Ignorance). All are symbols.

B. But if $T_A, T_R, F_A, F_R, U, C, G$ are subsets of [0, 1], then we get a *Seven Numerical-Valued Neutrosophic Logic*.

## *IV.1.6. n-Valued Logic*

*A. The n-Symbol-Valued Refined Neutrosophic Logic* [Smarandache, 1995].

In general:

$T$ can be split into many types of truths:

$T_1$, T2, ..., $T_p$,

and $I$ into many types of indeterminacies:

$I_1$, $I_2$, ..., $I_r$,

and $F$ into many types of falsities:

$F_1$, $F_2$, ..., $F_s$, where

all $p, r, s \geq 1$ are integers, and $p + r + s = n$.

All subcomponents $T_j$, $I_k$, $F_l$ are symbols for $j \in \{1, 2, \ldots, p\}$, $k \in \{1, 2, \ldots, r\}$, and $l \in \{1, 2, \ldots, s\}$.

If at least one $I_k = T_j \wedge F_l$ = contradiction, we get again the Extenics.

B. *The n-Numerical-Valued Refined Neutrosophic Logic*.

In the same way, but all subcomponents $T_j$, $I_k$, $F_l$ are not symbols, but subsets of [0,1], for all $j \in \{1, 2, \ldots, p\}$, all $k \in \{1, 2, \ldots, r\}$, and all $l \in \{1, 2, \ldots, s\}$.

If all sources of information that separately provide neutrosophic values for a specific subcomponent are independent sources, then in the

general case we consider that each of the sub-components $T_j$, $I_k$, $F_l$ is independent with respect to the others and it is in the non-standard set $]^-0,1^+[$. Therefore, per total we have for crisp neutrosophic value subcomponents $T_j$, $I_k$, $F_l$ that:

$$^-0 \leq \sum_{j=1}^{p} T_j + \sum_{k=1}^{r} I_k + \sum_{l=1}^{s} F_l \leq n^+ \qquad (1)$$

where of course $n = p + r + s$ as above. If there are some dependent sources (or respectively some dependent subcomponents), we can treat those dependent subcomponents together. For example, if $T_2$ and $I_3$ are dependent, we put them together as $^-0 \leq T_2 + I_3 \leq 1^+$.

The non-standard unit interval $]^-0,1^+[$., used to make a distinction between absolute and relative truth / indeterminacy / falsehood in philosophical applications, is replace for simplicity with the standard (classical) unit interval $[0,1]$ for technical applications.

For at least one $I_k$=$T_j \wedge F_l$= contradiction, we get again the Extenics.

## *IV.1.7. n-Valued Neutrosophic Logic Connectors*

### *1. n-Norm and n-Conorm defined on combinations of t-Norm and t-Conorm*

The n-norm is actually the neutrosophic conjunction operator, NEUTROSOPHIC AND ($\wedge_n$); while the n-conorm is the neutrosophic disjunction operator, NEUTROSOPHIC OR ($\vee_n$).

One can use the t-norm and t-conorm operators from the fuzzy logic in order to define the *n-norm* and respectively *n-conorm* in neutrosophic logic:

$$n-norm\left(\left(T_j\right)_{j=\{1,2,\ldots,p\}}, \left(I_k\right)_{k=\{1,2,\ldots,r\}}, \left(F_l\right)_{l=\{1,2,\ldots,s\}}\right) =$$

$$\begin{pmatrix} \left[t-norm\left(T_j\right)\right]_{j=\{1,2,\ldots,p\}}, \\ \left[t-conorm\left(I_k\right)\right]_{k=\{1,2,\ldots,r\}}, \\ \left[t-conorm\left(F_l\right)\right]_{l=\{1,2,\ldots,s\}} \end{pmatrix} \tag{2}$$

and

$$n-conorm\left(\left(T_j\right)_{j=\{1,2,\ldots,p\}}, \left(I_k\right)_{k=\{1,2,\ldots,r\}}, \left(F_l\right)_{l=\{1,2,\ldots,s\}}\right) =$$

$$\begin{pmatrix} \left[t-conorm\left(T_j\right)\right]_{j=\{1,2,\ldots,p\}}, \\ \left[t-norm\left(I_k\right)\right]_{k=\{1,2,\ldots,r\}}, \\ \left[t-norm\left(F_l\right)\right]_{l=\{1,2,\ldots,s\}} \end{pmatrix} \tag{3}$$

and then one normalizes if needed.

Since the n-norms/n-conorms, alike t-norms/t-conorms, can only approximate the inter-connectivity between two n-Valued Neutrosophic Propositions, there are many versions of these approximations.

For example, for the n-norm: the indeterminate (sub)components $I_k$ alone can be combined with the t-conorm in a pessimistic way [i.e. lower bound], or with the t-norm in an optimistic way [upper bound]; while for the n-conorm: the indeterminate (sub)components $I_k$ alone can be combined with the t-norm in a pessimistic way [i.e. lower bound], or with the t-conorm in an optimistic way [upper bound].

In general, if one uses in defining an n-norm/n-conorm for example the t-norm $min(x, y)$ then it is indicated that the corresponding t-conorm used be $max(x, y)$; or if the t-norm used is the product $x \cdot y$ then the corresponding t-conorm should be $x + y - x \cdot y$, and similarly if the t-norm used is $max\{0, x + y - 1\}$ then the corresponding t-conorm should be $min\{x + y, 1\}$, and so on.

Yet, it is still possible to define the n-norm and n-conorm using different types of t-norms and t-conorms.

*2. N-norm and n-conorm based on priorities*

For the n-norm we can consider the priority:

$T < I < F$,

where the subcomponents are supposed to conform with similar priorities, i.e.

$$T_1 < T_2 < \cdots < T_p < I_1 < I_2 < \cdots < I_r < F_1 < F_2 < \cdots < F_s \quad (4)$$

While for the n-conorm one has the opposite priorities:

$T > I > F$, or for the refined case:

$$T_1 > T_2 > \cdots > T_p > I_1 > I_2 > \cdots > I_r > F_1 > F_2 > \cdots > F_s \quad (5)$$

By definition $A < B$ means that all products between $A$ and $B$ go to $B$ (the bigger).

Let's say, one has two neutrosophic values in simple (nonrefined case):

$$(T_x, I_x, F_x) \quad (6)$$

and

$$(T_y, I_y, F_y) \quad (7)$$

Applying the n-norm to both of them, with priorities $T < I < F$, we get:

$$(T_x, I_x, F_x) \wedge_n (T_y, I_y, F_y) = \left( \begin{matrix} T_xT_y, T_xI_y + T_yI_x + \\ I_xI_y, T_xF_y + T_yF_x + I_xF_y + I_yF_x + F_xF_y \end{matrix} \right). \quad (8)$$

Applying the n-conorm to both of them, with priorities $T > I > F$, we get:

$$(T_x, I_x, F_x) \vee_n (T_y, I_y, F_y) = \left( \begin{matrix} T_xT_y + T_xI_y + T_yI_x + \\ T_xF_y + T_yF_x + I_xI_y + I_xF_y + I_yF_x, F_xF_y \end{matrix} \right). \quad (9)$$

In a lower bound (pessimistic) n-norm one considers the priorities $T < I < F$, while in an upper bound (optimistic) n-norm one considers the priorities $I < T < F$.

Whereas, in an upper bound (optimistic) n-conorm one considers $T > I > F$, while in a lower bound (pessimistic) n-conorm one considers the priorities $T > F > I$.

Various priorities can be employed by other researchers depending on each particular application.

## *IV.1.8. Particular Cases*

If in 6a) and b) one has all $I_k = 0, k = 1, 2, \ldots, r$, we get the *n-Valued Refined Fuzzy Logic*.

If in 6a) and b) one has only one type of indeterminacy, i.e. $k = 1$, hence $I_1 = I > 0$, we get the *n-Valued Refined Intuitionistic Fuzzy Logic*.

## *IV.1.9. Distinction between Neutrosophic Physics and Paradoxist Physics*

Firstly, we make a distinction between Neutrosophic Physics and Paradoxist Physics.

### *IV.1.9.1. Neutrosophic Physics*

Let <A> be a physical entity (i.e. concept, notion, object, space, field, idea, law, property, state, attribute, theorem, theory, etc.), <antiA> be the opposite of <A>, and <neutA> be their neutral (i.e. neither <A> nor <antiA>, but in between).

Neutrosophic Physics is a mixture of two or three of these entities <A>, <antiA>, and <neutA> that hold together.

Therefore, we can have neutrosophic fields, and neutrosophic objects, neutrosophic states, etc.

### *IV.1.9.2. Paradoxist Physics*

Neutrosophic Physics is an extension of Paradoxist Physics, since Paradoxist Physics is a combination of physical contradictories <A> and <antiA> only that hold together, without referring to their neutrality <neutA>. Paradoxist Physics describes collections of objects or states that are individually characterized by contradictory properties, or are characterized neither by a property nor by the opposite of that property, or are composed of contradictory sub-elements. Such objects or states are called paradoxist entities.

These domains of research were set up in the 1995 within the frame of neutrosophy, neutrosophic logic/ set/probability/statistics.

## *IV.1.10. n-Valued Refined Neutrosophic Logic Applied to Physics*

There are many cases in the scientific (and also in humanistic) fields that two or three of these items <A>, <antiA>, and <neutA> simultaneously coexist.

*Several Examples* of paradoxist and neutrosophic entities:

— anions in two spatial dimensions are arbitrary spin particles that are neither bosons (integer spin) nor fermions (half integer spin);

— among possible Dark Matter candidates there may be exotic particles that are neither Dirac nor Majorana fermions;

— mercury (Hg) is a state that is neither liquid nor solid under normal conditions at room temperature;

— non-magnetic materials are neither ferromagnetic nor anti-ferromagnetic;

— quark gluon plasma (QGP) is a phase formed by quasifree quarks and gluons that behaves neither like a conventional plasma nor as an ordinary liquid;

— unmatter, which is formed by matter and antimatter that bind together (F. Smarandache, 2004);

— neutral kaon, which is a pion and anti-pion composite (R. M. Santilli, 1978) and thus a form of unmatter;

— neutrosophic methods in General Relativity (D. Rabounski, F. Smarandache, L. Borissova, 2005);

— neutrosophic cosmological model (D. Rabounski, L. Borissova, 2011);

— neutrosophic gravitation (D. Rabounski);

— qubit and generally quantum superposition of states;

— semiconductors are neither conductors nor isolators;

— semi-transparent optical components are neither opaque nor perfectly transparent to light;

— quantum states are metastable (neither perfectly stable, nor unstable);

— neutrino-photon doublet (E. Goldfain);

— the "multiplet" of elementary particles is a kind of "neutrosophic field" with two or more values (E. Goldfain, 2011);

— a "neutrosophic field" can be generalized to that of operators whose action is selective. The effect of the neutrosophic field is somehow equivalent with the "tunneling" from the solid physics, or with the "spontaneous symmetry breaking" (SSB) where there is an internal symmetry which is broken by a particular selection of the vacuum state (E. Goldfain). Etc.

Many types of logics have been presented above. For the most general logic, the n-valued refined neutrosophic logic, we presented two classes of neutrosophic operators to be used in combinations of neutrosophic valued propositions in physics.

Similar generalizations are done for *n-Valued Refined Neutrosophic Set*, and respectively *n-Valued Refined Neutrosophic Probability*.

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**CHAPTER V**

# *V.1. Operations with Neutrosophic Numbers*

Let $<t_1, i_1, f_1>$ and $<t_2, i_2, f_2>$ be two neutrosophic numbers, and $\alpha \in \mathbb{R}$ be a real scalar number. Then one has:

## *V.1.1. Addition of Neutrosophic Numbers*

$$<t_1, i_1, f_1> \oplus <t_2, i_2, f_2> = <t_1 + t_2 - t_1 t_2, i_1 i_2, f_1 f_2> \quad (1)$$

## *V.1.2. Subtraction of Neutrosophic Numbers* *(Smarandache 2016, Ye 2017)*

$$<t_1, i_1, f_1> \ominus <t_2, i_2, f_2> = <\frac{t_1 - t_2}{1 - t_2}, \frac{i_1}{i_2,}, \frac{f_1}{f_2}>, \quad (2)$$

where

$$\frac{t_1 - t_2}{1 - t_2} = \begin{cases} 0, \text{if } t_1 < t_2, \text{or } t_1 = t_2; \\ \frac{t_1 - t_2}{1 - t_2}, \text{if } \frac{t_1 - t_2}{1 - t_2} \in [0,1]; \end{cases} \quad (3)$$

$$\frac{i_1}{i_2} = \begin{cases} \frac{i_1}{i_2}, \text{if} \frac{i_1}{i_2} \in [0,1]; \\ 1, \text{if } \frac{i_1}{i_2} > 1, \text{or } i_2 = 0; \end{cases} \quad (4)$$

$$\frac{f_1}{f_2} = \begin{cases} \frac{f_1}{f_2}, \text{if } \frac{f_1}{f_2} \in [0,1]; \\ 1, \text{if } \frac{f_1}{f_2} > 1, \text{or } f_2 = 0. \end{cases} \quad (5)$$

## *V.1.3. Multiplication of Neutrosophic Numbers*

$$<t_1, i_1, f_1> \odot <t_2, i_2, f_2>=$$
$$= \langle t_1 t_2, i_1 + i_2 - i_1 i_2, f_1 + f_2 - f_1 f_2 \rangle \quad (6)$$

## *V.1.4. Division of Neutrosophic Numbers*

(Smarandache 2016, Ye 2017)

$$\langle t_1, i_1, f_1 \rangle \oslash \langle t_2, i_2, f_2 \rangle = \langle \frac{t_1}{t^2}, \frac{i_1 - i_2}{1 - i_2}, \frac{f_1 - f_2}{1 - f_2} \rangle \quad (7)$$

where

$$\frac{t_1}{t_2} = \begin{cases} \frac{t_1}{t_2}, \text{if} \frac{t_1}{t_2} \in [0,1]; \\ 1, \text{if} \frac{t_1}{t_2} > 1, \text{or } t^2 = 0; \end{cases} \quad (8)$$

$$\frac{i_1 - i_2}{1 - i_2} = \begin{cases} 0, \text{if } i_1 < i_2, \text{or } i_1 = i_2; \\ \frac{i_1 - i_2}{1 - i_2}, \text{if } \frac{i_1 - i_2}{1 - i_2} \in [0,1]; \end{cases} \quad (9)$$

$$\frac{f_1 - f_2}{1 - f_2} = \begin{cases} 0, \text{if } f_1 < f_2, \text{or } f_1 = f_2; \\ \frac{f_1 - f_2}{1 - f_2}, \text{if } \frac{f_1 - f_2}{1 - f_2} \in [0,1]. \end{cases} \quad (10)$$

## *V.1.5. Scalar Multiplication of Neutrosophic Numbers*

For $\lambda > 0$, $\alpha \odot \langle t_1, i_1, f_1 \rangle = \langle t_1, i_1, f_1 \rangle \odot \alpha =$

$$= \langle 1 - (1 - t_1)^\lambda, i_1^\lambda, f_1^\lambda \rangle. \quad (11)$$

## *V.1.6. Power of Neutrosophic Numbers*

For $\lambda > 0$, $\langle t_1, i_1, f_1 \rangle^{\lambda} =$

$$= \langle t_1^{\lambda}, 1 - (1 - i_1)^{\lambda}, 1 - (1 - f_1)^{\lambda} \rangle. \qquad (12)$$

# *V.2. Addition of Multiple Single-Valued Neutrosophic Numbers*

For $h \in \{1, 2, ..., m\}$, let $N_h = (t_h, i_h, f_h)$ be single-valued neutrosophic numbers, with all

$t_h, i_h, f_h \in [0, 1]$.

$N_1 \oplus N_2 \oplus ... \oplus N_m = < t_1 \oplus t_2 \oplus ... \oplus t_m, i_1 \oplus i_2 \oplus ... \oplus i_m, f_1 \oplus f_2 \oplus ... \oplus f_m >$. (1)

For $t_1, t_2, ..., t_n$ as neutrosophic truth components of neutrosophic numbers, one has:

$t_1 \oplus t_2 = t_1 + t_2 - t_1t_2 = \{t_1 + t_2\} - \{t_1t_2\} = S_1 - S_2$. (2)

$(t_1 \oplus t_2) \oplus t_3 = (t_1 + t_2 - t_1t_2) \oplus t_3 = t_1 + t_2 + t_3 - t_1t_2 - t_1t_3 - t_2t_3 + t_1t_2t_3$

$= \{t_1 + t_2 + t_3\} - \{t_1t_2 + t_2t_3 + t_3t_1\} + \{t_1t_2t_3\} =$

$= S_1 - S_2 + S_3$. (3)

$(t_1 \oplus t_2 \oplus t_3) \oplus t_4 = (t_1 + t_2 + t_3 - t_1t_2 - t_1t_3 - t_2t_3 + t_1t_2t_3) \oplus t_4 = t_1 + t_2 + t_3 + t_4 - t_1t_2 - t_1t_3 - t_2t_3 - t_1t_4 - t_2t_4 - t_3t_4 + t_1t_2t_3 + t_1t_2t_4 + t_1t_3t_4 + t_2t_3t_4 - t_1t_2t_3t_4$

$= \{t_1 + t_2 + t_3 + t_4\} - \{t_1t_2 - t_1t_3 - t_2t_3 - t_1t_4 - t_2t_4 - t_3t_4\} + \{t_1t_2t_3 + t_1t_2t_4 + t_1t_3t_4 + t_2t_3t_4\} - \{t_1t_2t_3t_4\}$

$= S_1 - S_2 + S_3 - S_4$. (4)

And in general: (5)

$t_1 \oplus t_2 \oplus ... \oplus t_m = S_1 - S_2 + ... + (-1)^{k+1}S_k + ...$

$$+ (-1)^{m+1}S_m = \sum_{k=1}^{m}(-1)^{k+1}S_k = \sum_{k=1}^{m}(-1)^{k+1}\sum_{\{j_1,j_2,...,j_k\}} t_{j_1}t_{j_2}...t_{j_k}$$

and, for $1 \le k \le m$, one has $S_k = \sum_{\{j_1,j_2,...,j_k\}} t_{j_1}t_{j_2}...t_{j_k}$,

where $\{j_1, j_2, ..., j_k\}$ are permutations of *m* elements *{1, 2, ..., m}* taken by groups of *k* elements.

For $i_1$, $i_2$, ..., $i_n$ as neutrosophic indeterminacy components of neutrosophic numbers, one simply has:

$$i_1 \oplus i_2 \oplus ... \oplus i_m = i_1 i_2 \cdot ... \cdot i_m. \quad (6)$$

And similarly, for $f_1$, $f_2$, ..., $f_n$ as neutrosophic falsehood components of neutrosophic numbers, one simply has:

$$f_1 \oplus f_2 \oplus ... \oplus f_m = f_1 f_2 \cdot ... \cdot f_m. \quad (7)$$

Whence, putting all three neutrosophic components together, we get the general formula:

$$N_1 \oplus N_2 \oplus ... \oplus N_m = < \sum_{k=1}^{m}(-1)^{k+1}\sum_{\{j_1,j_2,...,j_k\}} t_{j_1}t_{j_2}...t_{j_k},$$

$$i_1 i_2 \cdot ... \cdot i_m,\ f_1 f_2 \cdot ... \cdot f_m >, \quad (8)$$

where $\{j_1, j_2, ..., j_k\}$ are permutations of *m* elements *{1, 2, ..., m}* taken by groups of *k* elements.

# *V.3. Interval-Valued Neutrosophic Number Operations*

We first define the following *Operators for Interval-Valued Neutrosophic Numbers*.

Let $S_1$ and $S_2$ be two intervals included in [0, 1].

Below "inf" means infimum and "sup" means supremum, while [ , ] or [ , ) or ( , ], or ( , ) mean interval.

## *V.3.1. Addition of Intervals*

$$S_1 + S_2 = [a, b], \quad (1)$$

where $a = \inf(S_1) + \inf(S_2)$, and $b = \sup(S_1) + \sup(S_2)$.

## *V.3.2. Multiplication of Intervals*

$$S_1 \times S_2 = [c, d] \quad (2)$$

where $c = \inf(S_1) \times \inf(S_2)$, and $d = \sup(S_1) \times \sup(S_2)$.

## *V.3.3. Subtraction of Intervals*

$$S_1 - S_2 = [\alpha, \beta], \quad (3)$$

where

$$\alpha = \begin{cases} \inf(S_1) - \sup(S_2), \inf(S_1) \geq \sup(S_2); \\ 0, otherwise. \end{cases}$$

$$\beta = \begin{cases} \sup(S_1) - \inf(S_2), \sup(S_1) \geq \inf(S_2); \\ 0, \textit{otherwise.} \end{cases}$$

## *V.3.4. Division of Intervals*

$$S_1 / S_2 = [\gamma, \delta], \tag{4}$$

where

$$\gamma = \begin{cases} \dfrac{\inf(S_1)}{\sup(S_2)}, \inf(S_1) \leq \sup(S_2); \\ 1, \textit{otherwise.} \end{cases}$$

$$\delta = \begin{cases} \dfrac{\sup(S_1)}{\inf(S_2)}, \sup(S_1) \leq \inf(S_2); \\ 1, \textit{otherwise.} \end{cases}$$

We can now straightforwardly generalize the single-valued neutrosophic number operations to *interval-valued neutrosophic number operations*.

Let $A(T_1, I_1, F_1)$ and $B(T_2, I_2, F_2)$ be two interval-valued neutrosophic numbers of the universe of discourse U, where their neutrosophic components $T_1, I_1, F_1, T_2, I_2, F_2$ are intervals included in the interval [0, 1].

All the below operations involving $T_1, I_1, F_1, T_2, I_2, F_2$ are additions, subtractions, multiplications, or divisions of intervals as defined above:

### *V.3.5. Addition of Interval-Valued Neutrosophic Numbers*

(which is actually like neutrosophic union):

$$A + B = (T_1, I_1, T_1) + (T_2, I_2, F_2) =$$

$$= (T_1 + T_2 - T_1T_2, I_1I_2, F_1F_2). \quad (5)$$

### *V.3.6. Multiplication of Interval-Valued Neutrosophic Numbers*

(which is actually like neutrosophic intersection):

$$A \times B = (T_1, I_1, T_1) \times (T_2, I_2, F_2) =$$

$$= (T_1T_2, I_1 + I_2 - I_1I_2, F_1 + F_2 - F_1F_2). \quad (6)$$

### *V.3.7. Subtraction of Interval-Valued Neutrosophic Numbers:*

$$A - B = (T_1, I_1, T_1) - (T_2, I_2, F_2) =$$

$$= ((T_1 - T_2)/(1 - T_2), I_1/I_2, F_1/F_2). \quad (7)$$

### *V.3.8. Division of Interval-Valued Neutrosophic Numbers:*

$$A / B = (T_1, I_1, F_1) / (T_2, I_2, F_2) =$$

$$= (T_1/T_2, (I_1-I_2)/(1-I_2), (F_1-F_2)/(1-F_2)). \quad (8)$$

*Remark:* The operations can straightforwardedly be extended from interval-valued to subunitary subset-valued neutrosophic components.

# *V.4. Operations with (t, i, f)-Neutrosophic Matrices*

One uses the previous operations with neutrosophic numbers in defining the operations with $(t,i,f)$-neutrosophic matrices. Let $A = [a_{jk}]_{jk}$ and $B = [b_{jk}]_{jk}$, j∈{1, 2, ..., $m$}, k∈{1, 2, ..., $n$}, for $m,n \geq 1$ be two $(t,i,f,)$-neutrosophic matrices of $m \times n$ size.

Let $C = [c_{kl}]_{kl}$, $l$∈{1, 2, ..., $p$}, for $p \geq 1$, be another matrix of $n \times p$ size. All elements $a_{jk}$ are neutrosophic numbers, of the form:

$a_{jk} = \langle\ t_{jk}^{a}, i_{jk}^{a}, f_{jk}^{a} \rangle$,

and similarly

$b_{jk} = \langle t_{jk}^{b}, i_{jk}^{b}, f_{jk}^{b} \rangle$, $c_{kl} = \langle t_{kl}^{c}, i_{kl}^{c}, f_{kl}^{c} \rangle$.

## *V.4.1. Addition of $(t,i,f)$-Neutrosophic Matrices*

$$A \oplus B = [a_{jk} \oplus b_{jk}]_{jk}. \qquad (1)$$

### *V.4.1.1. A More General Definition of Addition of $(t,i,f)$-Neutrosophic Matrices*

$$A \oplus B = [a_{jk} \underset{N}{\vee} b_{jk}\ ]_{jk}. \qquad (2)$$

where $\overset{\vee}{N}$ is any neutrosophic disjunction operator.

### *V.4.2. Substraction of $(t,i,f)$-Neutrosophic Matrices*

$$A \ominus B = [a_{jk} \ominus b_{jk}]_{jk}. \tag{3}$$

### *V.4.3. Scalar Multiplication of $(t,i,f)$-Neutrosophic Matrices*

$$\alpha \odot A = [\alpha \odot a_{jk}]_{jk}. \tag{4}$$

### *V.4.4. Multiplication of $(t,i,f)$-Neutrosophic Matrices*

$$A \otimes C = \bigoplus_{k=1}^{n} [a_{jk} \otimes c_{kl}]_{jl}, \tag{5}$$

which is a matrix of size $m \times p$.

#### *V.4.4.1. A More General Definition of Multiplication of $(t,i,f)$-Neutrosophic Matrices*

$$A \otimes C = \bigvee_{k=1}^{n} \overset{\vee}{N} \left[a_{jk} \overset{\wedge}{N} c_{jk}\right]_{jl}, \tag{6}$$

where $\overset{\wedge}{N}$ is any neutrosophic conjunction operator and $\overset{\vee}{N}$ any neutrosophic disjunction operator that is applied $n$ times, upon the summation index $k$ taken values from 1 to $n$.

### *V.4.4.2. Remark*

For the general definitions of addition and multiplication of $(t,i,f)$-neutrosophic matrices, the neutrosophic operators $\overset{\wedge}{N}$ and $\overset{\vee}{N}$ can be associated correspondingly, which is the most indicated procedure, i.e.:

| $\overset{\wedge}{N}$ | | $\overset{\vee}{N}$ |
|---|---|---|
| min/max/max | with | max/min/min |
| $x \cdot y$ (product) | with | $x + y - xy$ (sum) |
| Łukasiewicz max/min/min | with | Łukasiewicz min/max/max |
| other t-norm | with | other t-conorm |

or randomly, as *hybrid neutrosophic operators*, for example:

| $\overset{\wedge}{N}$ | | $\overset{\vee}{N}$ |
|---|---|---|
| min/max/max | with | $x + y - xy$ (sum) |
| $x \cdot y$ (product) | with | max/min/min |
| Łukasiewicz max/min/min | with | max/min/min |
| min/max/max | with | Łukasiewicz min/max/max |

and in general, any neutrosophic operator from the leftt column, with another operator from the right column.

## *V.5. Examples*

Let's have two $(t,i,f)$-neutrosophic matrices:

$$A = \begin{bmatrix} < 0.1, 0.6, 0.3 > & < 0.2, 0.4, 0.5 > \\ < 0.7, 0.1, 0.1 > & < 0.6, 0.2, 0.8 > \end{bmatrix} \text{ and}$$

$$B = \begin{bmatrix} < 0.9, 0.2, 0.1 > & < 0.5, 0.5, 0.4 > \\ < 0.6, 0.3, 0.1 > & < 0.7, 0.2, 0.2 > \end{bmatrix}$$

a)

### *V.5.1. Addition*

Let's compute

$$A \oplus B = \begin{bmatrix} d_{11} & d_{12} \\ d_{21} & d_{22} \end{bmatrix} =$$

$$\begin{bmatrix} < 0.91, 0.12, 0.03 > & < 0.60, 0.20, 0.20. > \\ < 0.88, 0.03, 0.01 > & < 0.88, 0.04, 0.16 > \end{bmatrix};$$

$$d_{11} = < 0.1, 0.6, 0.3 > \oplus < 0.9, 0.2, 0.1 > =$$

$$< 0.1 + 0.9 - 0.1(0.9), 0.6(0.2), 0.3(0.1)$$

$$> = < 0.91, 0.12, 0.03 >,$$

and similary one computs $d_{12}$, $d_{21}$ and $d_{22}$ .

### *V.5.2. Multiplication*

$$A \otimes B = \begin{bmatrix} g^{11} & g^{12} \\ g^{21} & g^{22} \end{bmatrix} =$$

$$= \begin{bmatrix} < 0.1020, 0.3944, 0.703 > & < 0.1830, 0.4160, 0.3944 > \\ < 0.7632, 0.1012, 0.1538 > & < 0.6230, 0.1980, 0.3864 > \end{bmatrix}$$

$g_{11}$=<0.1,0.6,0.3>⊗<0.9,0.2,0.1>⊕<0.2,0.4,0.5>⊗
<0.6,0.3,0.1> = <0.1(0.9),0.6+0.2-0.6(0.2),0.3+
0.1-0.3(0.1)>⊕<0.2(0.6),0.4+0.3-0.4(0.3),0.1+
0.1-0.1(0.1)>=<0.09,0.68,0.37>⊕<0.12,0.58,0.19>
=<0.09+0.12-0.09(0.12),0.68(0.58),0.37(0.19)>
=<0.1020,0.3944,0.0703>;

$g_{12}$=<0.1,0.6,0.3>⊗<0.5,0.5,0.4>⊕<0.2,0.4,0.6>⊗
<0.7,0.2,0.2>=<0.05,0.80,0.58>⊕<0.14,0.52,0.68>
=<0.1830,0.4160,0.3944>;

$g_{21}$=<0.7,0.1,0.1>⊗<0.9,0.2,0.1>⊕<0.6,0.2,0.8>
⊗<0.6,0.3,0.1>=<0.63,0.28,0.19>⊕<0.36,0.44,0.82>
=<0.7632,0.1012,0.1558>;

$g_{22}$=<0.7,0.1,0.1>⊗<0.5,0.5,0.4>⊕<0.6,0.2,0.8>
⊗<0.7,0.2,0.2>=<0.35,0.55,0.46>⊕<0.42,0.36,0.84>
=<0.6230,0.1980,0.3864>.

b) Let's do the addition and multiplication of $(t, i, f)$-neutrosophic matrices using max/min for $\overset{\wedge}{N}$ and $\overset{\vee}{N}$ operators:

$<t_1,i_1,f_1> \overset{\vee}{N} <t_2,i_2,f_2>$=

$=<\max\{t_1,t_2\}, \min\{i_1,i_2\}, \min\{f_1,f_2\}>$

and

$<t_1,i_1,f_1>\overset{\wedge}{_N}<t_2,i_2,f_2>=$

$=<\min\{t_1,t_2\}, \max\{i_1,i_2\}, \max\{f_1,f_2\}>$

*V.5.3. Addition*

$$A \oplus B = \begin{bmatrix} r_{11} & r_{12} \\ r_{21} & r_{22} \end{bmatrix}$$

$$= \begin{bmatrix} < 0.9,0.2,0.1 > & < 0.5,0.4,0.4 > \\ < 0.7,0.1,0.1 > & < 0.7,0.2,0.2 > \end{bmatrix};$$

$r_{11} = <0.1,0.6,0.3>\overset{\vee}{_N}<0.9,0.2,0.1>=$

$=<\max\{0.1,0.9\}, \min\{0.6,0.2\}, \min\{0.3,0.1\}>$

$= <0.9,0.2,0.1>;$

similary for $r_{12}$, $r_{21}$ and $r_{22}$.

*V.5.4. Multiplication*

$$A \otimes B = \begin{bmatrix} h_{11} & h_{12} \\ h_{21} & h_{22} \end{bmatrix}$$

$$= \begin{bmatrix} < 0.2,0.4,0.3 > & < 0.2,0.4,0.4 > \\ < 0.7,0.2,0.1 > & < 0.7,0.2,0.5 > \end{bmatrix};$$

$h_{11} = <0.1,0.6,0.3>\overset{\wedge}{_N}<0.9,0.2,0.1>\overset{\vee}{_N}$

$<0.2,0.4,0.5>\overset{\wedge}{_N}<0.6,0.3,0.1>=$

$=<\min\{0.1,0.9, \max\{0.6,0.2\}, \max\{0.3,0.1\}>\overset{\vee}{_N}$

$<\min\{0.2,0.6\}, \max\{0.4,0.3\}, \max\{0.5,0.1\}>=$

$=<0.1,0.6,0.3>\overset{\vee}{}_{N}<0.2,0.4,0.5>=<\max\{0.1,0.2\},$

$\min\{0.6,0.4\}, \min\{0.3,0.5\}>=<0.2,0.4,0.3>;$

$h_{12} = <0.1,0.6,0.3>\overset{\wedge}{}_{N}<0.5,0.5,0.4>$

$\overset{\vee}{}_{N}<0.2,0.4,0.5>\overset{\wedge}{}_{N}<<0.7,0.2,0.2> =$

$= <0.1,0.6,0.4>\overset{\vee}{}_{N}<0.2,0.4,0.5>=<0.2,0.4,0.4>;$

$h_{21} = <0.7,0.1,0.1>\overset{\wedge}{}_{N}<0.9,0.2,0.1>$

$\overset{\vee}{}_{N}<0.6,0.2,0.8>\overset{\wedge}{}_{N}<0.6,0.3,0.1> =$

$<0.7,0.2,0.1>\overset{\vee}{}_{N}<0.6,0.3,0.8> = <0.7,0.2,0.1>;$

$h_{22} = <0.7,0.1,0.1>\overset{\wedge}{}_{N}<0.5,0.5,0.4>$

$\overset{\vee}{}_{N}<0.6,0.2,0.8>\overset{\wedge}{}_{N}<0.7,0.2,0.2> =$

$= <0.5,0.5,0.5>\overset{\vee}{}_{N}<0.7,0.2,0.8> = <0.7,0.2,0.5>.$

c) Let's do the addition and multiplication of $(t,i,f)$-neutrosophic matrices using Łukasiewicz operators, which is very rough:

$<t_1,i_1,f_1>\overset{\vee}{}_{N}<t_2,i_2,f_2>=<\min\{t_1+t_2,1\},$

$\max\{i_1+i_2-1,0\}, \max\{f_1+f_2-1,0\}>;$

$<t_1,i_1,f_1>\overset{\wedge}{}_{N}<t_2,i_2,f_2>=<\max\{t_1+t_2-1,0\},$

$\min\{i_1+i_2,1\}$, $\min\{f_1+f_2,1\}>$.

### *V.5.4. Addition*

$$A \oplus B = \begin{bmatrix} s_{11} & s_{12} \\ s_{21} & s_{22} \end{bmatrix} = \begin{bmatrix} < 1,0,0 > & < 0.7,0,0 > \\ < 1,0,0 > & < 1,0,0 > \end{bmatrix};$$

$s_{11}$ = <0.1,0.6,0.3> $\overset{\vee}{N}$ <<0.9,0.2,0.1> =

<min{0.1+0.9,1}, max{0.6+0.2-1,0},

max{0.3+0.1-1,0}> = <1, 0, 0>;

similary for $s_{12}$, $s_{21}$ and $s_{22}$ .

### *V.5.5. Multiplication*

$$A \otimes B = \begin{bmatrix} u_{11} & u_{12} \\ u_{21} & u_{22} \end{bmatrix}$$

$$= \begin{bmatrix} < 0,0.5,0 > & < 0,0.6,0.4 > \\ < 0.2,0,0.1 > & < 0.7,0,0.57 > \end{bmatrix};$$

$u_{11}$ = <0.1,0.6,0.3> $\overset{\wedge}{N}$ <0.9,0.2,0.1> $\overset{\vee}{N}$ <0.2,0.4,0.5>

$\overset{\wedge}{N}$ <0.6,0.3,0.1> = <max{0.1+0.9-1,0},

min{0.6+0.2,1}, min{0.3+0.1,1}> $\overset{\vee}{N}$

<max{0.2+0.6-1,0}, min{0.4+0.3,1},

min{0.5+0.1,1}> = <0,0.8, 0.4

> $\overset{\vee}{N}$ <0,0.7,0.6>=<min{0+0,1},

max{0.8+0.7-1,0}, max{0.4+0.6-1,0}> = <0,0.5,0>;

$u_{12} = \langle 0.1,0.6,0.3\rangle \overset{\wedge}{N} \langle 0.5,0.5,0.4\rangle$

$\overset{\vee}{N} \langle 0.2,0.4,0.5\rangle \overset{\wedge}{N} \langle 0.7,0.2,0.2\rangle =$

$= \langle 0,1,0.7\rangle \overset{\vee}{N} \langle 0,0.6,0.7\rangle = \langle 0,0.6,0.4\rangle;$

$u_{21} = \langle 0.7,0.1,0.1\rangle \overset{\wedge}{N} \langle 0.9,0.2,0.1\rangle \overset{\vee}{N}$

$\langle 0.6,0.2,0.8\rangle \overset{\wedge}{N} \langle 0.6,0.3,0.1\rangle =$

$\langle 0.6,0.3,0.2\rangle \overset{\vee}{N} \langle 0.2,0.5,0.9\rangle = \langle 0.8,0,0.1\rangle;$

$u_{22} = \langle 0.7,0.1,0.1\rangle \overset{\wedge}{N} \langle 0.5,0.5,0.4\rangle \overset{\wedge}{N}$

$\langle 0.6,0.2,0.8\rangle \overset{\wedge}{N} \langle 0.7,0.2,0.2\rangle =$

$= \langle 0.2,0.6,0.5\rangle \overset{\wedge}{N} \langle 0.3,0.4,1\rangle = \langle 0.7,0,0.5\rangle.$

**CHAPTER VI**

## *VI.1. Neutrosophic Hybrid Operators*

The neutrosophic operators, based on single-value fuzzy *t*-norm $\overset{\wedge}{}_{F}$ and fuzzy *t*-conorm $\overset{\vee}{}_{F}$ may be diversified in a hybrid way.

By $\overset{\wedge}{}_{F}$ and $\overset{\vee}{}_{F}$ we mean fuzzy intersection and fuzzy union, while by $\overset{\wedge}{}_{N}$ and $\overset{\vee}{}_{N}$ of course we mean neutrosophic intersection and neutrosophic union respectively.

Let's list on two columns $\overset{\wedge}{}_{F}$ and $\overset{\vee}{}_{F}$ below the most common ones:

| Fuzzy *t*-norm $\left(\overset{\wedge}{}_{F}\right)$ | Fuzzy *t*-conorm $\left(\overset{\vee}{}_{F}\right)$ |
|---|---|
| $x\overset{\wedge_1}{}_{F}y=\min\{x,y\}$ | $x\overset{\vee_1}{}_{F}y=\max\{x,y\}$ |
| $x\overset{\wedge_2}{}_{F}y=x\cdot y$ | $x\overset{\vee_2}{}_{F}y=x+y-xy$ |
| $x\overset{\wedge_3}{}_{F}y=\max\{x+y-1,0\}$ | $x\overset{\vee_3}{}_{F}y=\min\{x+y,1\}$ |
| *Others* | *Others* |

The most used others single-valued neutrosophic operators, based on these, are:

$$<t_1,i_1,f_1>\overset{\wedge}{}_{N}<t_2,i_2,f_2> = <t_1\overset{\wedge}{}_{F}t_2,i_1\overset{\vee}{}_{F}i_2,f_1\overset{\vee}{}_{F}f_2>, \qquad (1)$$

$$<t_1,i_1,f_1> \overset{\vee}{}_N <t_2,i_2,f_2> = <t_1 \overset{\vee}{}_F t_2, i_1 \overset{\wedge}{}_F i_2, f_1 \overset{\wedge}{}_F f_2>. \qquad (2)$$

The *neutrosophic implication* $A \underset{N}{\rightarrow} B$, where $A < t_1, i_1, f_1 >$ and $B < t_2, i_2, f_2 >$, can be transformed, as in classical logic, to $({}_N\neg A) \overset{\vee}{}_N B$, where ${}_N\neg$ is the neutrosophic negation, and, similary, the *neutrosophic equivalence* $A \underset{N}{\leftrightarrow} B$ transformed as in classical logic again into two neutrosophic implications: $(A \underset{N}{\rightarrow} B) \overset{\wedge}{}_N ( B \underset{N}{\rightarrow} A)$, which becomes:

$$\left[({}_N\neg A) \overset{\vee}{}_N B\right] \overset{\wedge}{}_N \left[({}_N\neg B) \overset{\vee}{}_N A\right] \qquad (3)$$

that is an expression depending on $\overset{\wedge}{}_N$ and $\overset{\vee}{}_N$ mostly (besides the neutrosophic negation ${}_N\neg$).

## *VI.1.1. Neutrosophic Hybrid Intersection*

$$< t_1, i_1, f_1 > \overset{H}{\underset{N}{\wedge}} < t_2, i_2, f_2 > =$$

$$= < t_1 \overset{\wedge_j}{}_F t_2, i_1 \overset{\vee_k}{}_F i_2, f_1 \overset{\vee_l}{}_F f_2 >, \qquad (4)$$

where $j, k, l \in \{1, 2, 3\}$. «H» stands for «hybrid». There are $3 \times 3 \times 3 = 27$ possibilites; among them 26 neutrosophic hybrid intersection operators.

That means, one can take any fuzzy intersection operator, from the first column, for the first neutrosophic component $t_1 \overset{\wedge}{F} t_2$; and any fuzzy union operator, from the second column, for the neutrosophic component $i_1 \overset{\vee_k}{F} i_2$; and similary, any fuzzy union operator, from the second column, for the third neutrosophic component $f_1 \overset{\vee_l}{F} f_2$.

Let's see an example:

$$< t_1, i_1, f_1 > \overset{H}{\underset{N}{\wedge}} < t_2, i_2, f_2 > =$$

$$<t_1 \overset{\vee_2}{F} t_2,\ i_1 \overset{\vee_3}{F} i_2\ ,\ f_1 \overset{\vee_2}{F} f_2> =$$

$$<t_1 \cdot t_2,\ \min\{x+y, 1\},\ f_1 + f_2 - f_1 f_2 >. \qquad (5)$$

## *VI.1.2. Neutrosophic Hybrid Union*

$$< t_1, i_1, f_1 > \overset{H}{\underset{N}{\vee}} < t_2, i_2, f_2 > =$$

$$< t_1 \overset{\vee_j}{F} t_2, i_1 \overset{\wedge_k}{F} i_2\ , f_1 \overset{\wedge_l}{F} f_2 >, \qquad (6)$$

where $j, k, l \in \{1, 2, 3\}$, and in the same way there are 27 possibilities, among them 26 neutrosophic hybrid union operators.

An example:

$$< t_1, i_1, f_1 > \overset{H}{\underset{N}{\vee}} < t_2, i_2, f_2 \geq$$

$$< t_1 \overset{\vee_1}{\underset{F}{}} t_2, i_1 \overset{\wedge_1}{\underset{F}{}} i_2, f_1 \overset{\wedge_3}{\underset{F}{}} f_2 >=$$

$$\langle \max\{t_1, t_2\}, \min\{i_1, i_2\}, \max\{f_1 + f_2 - 1, 0\} \rangle. \qquad (7)$$

## *VI.1.3. Neutrosophic Hybrid Implication*

Just replacing $\overset{\vee}{\underset{N}{}}$ by $\overset{H}{\underset{N}{\vee}}$ into its formula, and we get:

$$({}_N\neg A) \overset{H}{\underset{N}{\vee}} B \qquad (8)$$

or

$$({}_N\neg < t_1, i_1, f_1 >) \overset{H}{\underset{N}{\vee}} < t_2, i_2, f_2 >. \qquad (9)$$

The same number of possibilities (27), and same number of neutrosophic hybrid implication operators (26).

## *VI.1.4. Neutrosophic Hybrid Equivalence*

It has the formula:

$$\left[ ({}_N\neg A) \overset{H}{\underset{N}{\vee}} B \right] \overset{H}{\underset{N}{\wedge}} \left[ ({}_N\neg B) \overset{H}{\underset{N}{\vee}} A \right] \qquad (10)$$

or

$$\left[ \left( {}_{N}\neg < t_1, i_1, f_1 > \right) \underset{N}{\overset{H}{\vee}} < t_2, i_2, f_2 > \right] \underset{N}{\overset{H}{\wedge}} \left[ \left( {}_{N}\neg < t_2, i_2, f_2 > \right) \underset{N}{\overset{H}{\vee}} < t_1, i_1, f_1 > \right]. \qquad (11)$$

There are $27^3$ =19,683 possibilities, only one being non-hybrid.

However, if we take the two $\underset{N}{\overset{H}{\vee}}$ defined in the same way, then there are $27^2$ = 729 possibilities.

**CHAPTER VII**

# *VII.1. Neutrophic Triplets*

Let $\mathcal{U}$ be a universe of discourse, and $(N, *)$ a set from $\mathcal{U}$, endowed with a well-defined binary law $*$ (groupoid).

## *VII.1.1. Definition of Neutrosophic Triplet*

An element $a \in N$ forms a *neutrophic triplet* if there exist some *neutral element(s)* of $a$, denoted $neut(a) \in N$, where $neut(a)$ is different from the classical unitary element of $N$ with respect to the law $*$ (if any), such that

$$a * neut(a) = neut(a) * a = a \qquad (1)$$

and if there exist some *opposite element(s)* of $a$, denoted $anti(a) \in N$, such that

$$a * anti(a) = anti(a) * a = neut(a). \qquad (2)$$

The triplet $< a, neut(a), anti(a) >$ is called a neutrosophic triplet.

## *VII.1.2. Example of Neutrosophic Triplet*

*neut(a)* has to be different from the classical unit element when we select the neutrals, which

means that among the *neut(a)*'s we take all except the classical unit element.

| * | **1** | **2** |
|---|---|---|
| **1** | 1 | 2 |
| **2** | 2 | 2 |

The set *( {1, 2}, * )* is a groupoid with classical unit element "*1*". Then *(2, 2, 2)* is the only neutrosophic triplet herein.

We do not take *(1, 1, 1)* as a neutrosophic triplet, since "*1*" is a classical groupoid unit.

## *VII.1.3. Definition of Neutrosophic Triplet Strong Set (or Neutrosophic Triplet Set)*

The groupoid *(N, *)* is called a neutrosophic triplet set if for any $a \in N$ there exist some neutral of *a*, denoted $neut(a) \in N$, different from the classical algebraic unitary element (if any), and some opposite of *a*, called $anti(a) \in N$.

## *VII.1.4. Example of Neutrosophic Triplet Strong Set*

| * | **1** | **2** |
|---|---|---|
| **1** | 2 | 1 |
| **2** | 1 | 1 |

The set *( {1,2}, * )* is a groupoid without classical unit element.

Then *<1, 2, 1>* and *<2, 1, 2>* are neutrosophic triplets.

The neutrosophic triplet strong set is *N = {1, 2}.*

## *VII.1.5. Definition of Neutrosophic Triplet Weak Set*

The groupoid *(N, *)* is called a neutrosophic triplet weak set if for any $a \in N$ there exist a neutrosophic triplet set *<b, neut(b), anti(b)>* included in *N* such that: $a = b$ or $a = neut(b)$ or $a = anti(b)$.

## *VII.1.6. Theorem*

Any neutrosophic triplet strong set is a neutrosophic triplet weak set, but not conversely.

*Proof.*

Let *(N, *)* be a neutrosophic triplet strong set. If $a \in N$, then *<a, neut(a), anti(a)>* is also included in *N*, therefore there exists a neutrosophic triplet in *N* that includes *a*, whence *N* is a neutrosophic triplet weak set.

Conversely we prove by using a counter-example.

Let $Z_3 = \{0, 1, 2\}$, embedded with the multiplication $\times$ modulo $3$, which is a well-defined law.

The classical unitary element in $Z_3$ is $1$.

$(Z_3, \times)$ is a neutrosophic triplet weak set, since the neutrosophic triplets formed in $Z_3$ with respect to the law $\times$ contain all elements $0, 1, 2$: $<0, 0, 0>$, $<0, 0, 1>$, and $<0, 0, 2>$.

But $(Z_3, \times)$ is not a neutrosophic triplet strong set, since, for example, for $2 \in Z_3$ there is no $neut(2) \neq 1$ and no $anti(2)$.

## *VII.1.7. Definition of Neutrosophic Triplet Strong Group (or Neutrosophic Triplet Group)*

Let $(N, *)$ be a neutrosophic triplet strong set. Then $(N, *)$ is called a neutrosophic triplet strong group (or neutrosophic triplet group), if the following classical axioms are satisfied.

1) $(N, *)$ is well-defined, i.e. for any $x, y \in N$ one has $x*y \in N$.

2) *(N, *)* is associative, i.e. for any *x, y, z* ∈ *N* one has *x*(y*z) = (x*y)**z.

*NTSG*, in general, is not a group in the classical way, because it may not have a classical unitary element, nor classical inverse elements. The neutrosophic neutrals replace the classical unitary element, and the neutrosophic opposites replace the classical inverse elements.

## *VII.1.8. Example of Neutrosophic Triplet Strong Group*

Let (N, *) be a neutrosophic triplet group, defined by a Cayley Table:

| * | ***a*** | ***b*** |
|---|---|---|
| ***a*** | *a* | *a* |
| ***b*** | *a* | *b* |

which has the following neutrosophic triplets: <a, a, a>, <a, a, b>, <b, b, b>. Therefore, if *a = neut(a),* one has *anti(a) = a* but also *anti(a) = b*.

### *VII.1.9. Proposition*

A neutrosophic triplet weak group does not exist, since there are elements that do not have neutrals or opposites.

### *VII.1.10. Definition of Neutrosophic Perfect Triplet*

A neutrosophic triplet $< a, b, c >$, for $a, b, c \in \mathbb{N}$, is called a *neutrosophic perfect triplet* if both

$< c, b, a >$ and $< b, b, b >$ are also neutrosophic triplets.

### *VII.1.11. Definition of Neutrosophic Imperfect Triplet*

A neutrosophic triplet $< a, b, c >$, for $a, b, c \in \mathbb{N}$, is called a *neutrosophic imperfect triplet* if at least one of $< c, b, a >$ or $< b, b, b >$ is not a neutrosophic triplet(s).

### *VII.1.12. Examples of Neutrosophic Perfect and Imperfect Triplets*

Let $A = \{0, 1, 2, \ldots, 9\}$, endowed with the classical multiplication law $(\times)$ *modulo 10*, which is well-

defined on *A*, with classical unitary element *1*. *A* is a neutrosophic triplet weak commutative set.

*B = {4, 6}* is a neutrosophic triplet strong subset of *A*, since *B ⊂ A*, and also a neutrosophic triplet strong group, whose neutrosophic triplets in *B* are *<4, 6, 4>* and *<6, 6, 6>*. *B* is generated by *{4}*, since

$4^1 = 4$, $4^2 = 6$, $4^3 = 4$, and so on *modulo 10*, therefore *B* is a neutro-cyclic triplet strong group. Similarly *C = {6}* is a neutro-cyclic triplet strong subgroup of *B*, since *C ⊂ B*, generated by *6*, whose single neutrosophic triplet is *<6, 6, 6>*.

We have the following *23* neutrosophic triplets in *A*:

| 6 *Neutrosophic Perfect Triplets* | 17 *Neutrosophic Imperfect Triplets* |
|---|---|
| *<0,0,0>* | *<0,0,1>, <0,0,2>, ..., <0,0,9>;* |
| *<5,5,5>* | *<5,5,1>, <5,5,3>, <5,5,7>, <5,5,9>;* |
| *<6,6,6>* | *<6,6,1>;* |
| *<2,6,8>* | *<2,6,3>;* |
| *<4,6,4>* | *<4,6,9>;* |
| *<8,6,2>* | *<8,6,7>.* |

Let's show the computation of several of them.

We take a perfect one $< 2,6,8 >$, $a = 2$, and show that $neut(2) = 6$ and $anti(2) = 8$.

*Proof*

$2 * neut(2) = neut(2) * 2 = 2$ means that $2 \times 6 = 6 \times 2 = 12 = 2\ (mod 10)$;

and $2 * anti(2) = anti(2) * 2 = neut(2)$ means that $2 \times 8 = 8 \times 2 = 16 = 6\ (mod 10)$.

Its reciprocal $< 8,6,2 >$ and $< 6,6,6 >$ are also neutrosophic triplets.

Let's take an imperfect triplet $< 4,6,9 >$, with $a = 4$, and show that $neut(4) = 6$, $anti(4) = 9$.

*Proof*

$4 * neut(4) = neut(4) * 4 = 4$ means that $4 \times 6 = 6 \times 4 = 24 = 4\ (mod 10)$;

and $4 \times anti(4) = anti(4) \times 4 = neut(4)$ means that $4 \times 9 = 9 \times 4 = 36 = 6\ (mod 10)$.

Its reciprocal $< 9,6,4 >$ is not a neutrosophic triplet, because

$9 \times 6 = 6 \times 9 = 54 = 4 \neq 9\ (mod 10)$,

hence $neut(9) \neq 6$;

but $< 6,6,6 >$ is a neutrosophic triplet.

The other neutrosophic triplets can be checked in the same way.

It should be remarked that the below four triplets:

$< 1, ①, 1 >< 3, ①, 7 >< 7, ①, 3 >< 9, ①, 9 >$,

although they verify the neut-axiom and anti-axiom, are excluded from the neutrosophic triplets since their neutral (1) is the same as the set's classical unitary element (1).

## *VII.1.13. Definition of Neutrosophic Triplet Relationship of Equivalence*

A neutrosophic triplet relationship of equivalence on a neutrosophic triplet (strong or weak) set (N, *) is a relationship $\mathcal{E}$ defined as follows.

$\forall\, a, b, c \in N$, one has the following axioms:

1) $a\mathcal{E}a$;
2) if $a\mathcal{E}b$ then $b\mathcal{E}a$;
3) if $a\mathcal{E}b$ and $b\mathcal{E}c$, then $a\mathcal{E}c$.

## *VII.1.14. Example of Neutrosophic Triplet Relationship of Equivalence*

Let $\mathcal{E}$ be a neutrosophic triplet relationship of equivalence on a neutrosophic triplet (strong or weak) set *(N, *),* defined as:

$\forall\, a, b \in N, a\mathcal{E}b \Leftrightarrow neut(a) = neut(b)$.

It can be easily proven that $\mathcal{E}$ is an equivalence, since:

4) $a\mathcal{E}a \Leftrightarrow neut(a) = neut(a)$;
5) *If* $a\mathcal{E}b$ *then* $b\mathcal{E}a$, *or if* $neut(a) = neut(b)$ *then* $neut(b) = neut(a)$;
6) *If* $a\mathcal{E}b$ *and* $b\mathcal{E}c$, *then* $a\mathcal{E}c$, *or if* $neut(a) = neut(b)$ *and* $neut(b) = neut(c)$, *then* $neut(a) = neut(c)$.

The number of neutrosophic triplet classes of equivalence, with respect to $\mathcal{E}$ on the previous *(A, *)* neutrosophic triplet weak set, is three:

$\{\hat{0}\} = \{< 0,0,0 >, < 0,0,1 >, < 0,0,2 >, \ldots, < 0,0,9 >\}$;

$\{\hat{5}\} = \{< 5,5,5 >, < 5,5,1 >, < 5,5,3 >, < 5,5,7 >,$
$< 5,5,9 >\}$;

and

$\{\hat{6}\} = \{< 2,6,8 >, < 2,6,3 >, < 4,6,4 >, < 4,6,9 >,$
$< 6,6,6 >, < 6,6,1 >, < 8,6,2 >, < 8,6,7 >\}$.

## *VII.1.15. Example of Neutrosophic Perfect and Imperfect Triplets*

$\mathbb{Z}_4 = \{0,1,2,3\}$, with the classical multiplication $(\times)$ *modulo* $4$; classical unitary element *1*. $\mathbb{Z}_4$ is a neutrosophic triplet commutative weak set.

| *Neutrosophic Perfect Triplet* | *Neutrosophic Imperfect Triplets* |
|---|---|
| <0,0,0> | <0,0,1>, <0,0,2>, <0,0,3>. |

## *VII.1.16. Example of Neutrosophic Perfect and Imperfect Triplets*

$\mathbb{Z}_6 = \{0, 1, 2, 3, 4, 5\}$, with classical multiplication $(\times)$ *modulo* $6$; classical unitary element *1*. $\mathbb{Z}_6$ is a neutrosophic triplet commutative weak set.

| *Neutrosophic Perfect Triplet* | *Neutrosophic Imperfect Triplets* |
|---|---|
| <0,0,0> | <0,0,1>, ..., <0,0,5>; |
| <2,4,2> | <2,4,5>; |
| <3,3,3> | <3,3,1>, <3,3,5>; |
| <4,4,4> | <4,4,1>. |

## *VII.1.17. Example of Non-Associative Law*

Let $\mathbb{Z}_5 = \{0, 1, 2, 3, 4\}$ endowed with the law $*$ defined by $a * b = 2a + 2b \pmod 5$, for any $a, b \in \mathbb{Z}_5$. This law is well-defined, non-associative, commutative, non-unitary, with zero-divisors.

For a neutrosophic triplet $< a, neut(a), anti(a) >$ to exist, it is necessary that $neut(a)$ depends on «$a$», because if $neut(a)$ was independent of «$a$», then it would be the classical unitary element. Whence:

$$a * neut(a) = 2a + 2neut(a) = a \pmod 5 \qquad (3)$$

or

$$2 \cdot neut(a) = -a \pmod 5, \qquad (4)$$

or

$$neut(a) = 2^{-1}(-a) = -3a = 2a \pmod 5. \qquad (5)$$

We do not need to check $neut(a) * a$ any further, since the law is commutative.

Therefore $neut(a) = 2a \pmod 5$ for any $a \in \mathbb{Z}_5$.

Let's find the $anti(a)$:

$$a * anti(a) = neut(a) \qquad (6)$$

or

$$a * anti(a) = 2a \qquad (7)$$

or

$$2a + 2 \cdot anti(a) = 2a \pmod 5 \tag{8}$$

or

$$2 \cdot anti(a) = 0 \pmod 5 \tag{9}$$

or

$$anti(a) = 0 \pmod 5, \tag{10}$$

since *2* and *5* are relatively prime,

or

$$anti(a) = 0, \tag{11}$$

because $anti(a) \in \mathbb{Z}_5$.

The general neutrosophic triplets are:

$< a, 2a, 0 > \pmod 5$,

where $a \in \{0,1,2,3,4\}$, whence one has the following neutrosophic imperfect triplets:

<0,0,0>, for a = 0;

<1,2,0>, for a = 1;

<2,4,0>, for a = 2;

<3,1,0>, for a = 3;

<4,3,0>, for a = 4.

Since the law is not associative, then if $< a, b, c >$ is a neutrosophic triplet it does not involve, in general, that $< c, b, a >$ or $< b, b, b >$ are neutrosophic triplets.

Also, because the law is commutative, we need to verify only one part of the axioms of $neut(a)$ and $anti(a)$, respectively.

## *VII.1.18. Definition of Neutrosophic Enemy of Itself*

Let *(N, *)* be a neutrosophic triplet (strong or weak) set. We say that the elements $a \in N$ is a neutrosophic enemy of itself if $a \in \{anti(a)\}$.

## *VII.1.19. Definition of Two Neutrosophic Friends*

Let *(N, *)* be a neutrosophic triplet (strong or weak) set.

We say that the elements $a_1, a_2 \in N$ are neutrosophic friends, if:

1) There exist $\{anti(a_1)\} \neq \emptyset$ and $\{anti(a_2)\} \neq \emptyset$ in the case when N is a neutrosophic triplet weak set [because in the case of neutrosophic triplet strong set they exist by definition], such that:

$\{anti(a_1)\} \cap \{anti(a_2)\} \neq \emptyset$,

i.e. $a_1, a_2$ have common enemies;

2) Also, $a_1 \notin \{anti(a_2)\}$ and $a_2 \notin \{anti(a_1)\}$,

i.e. $a_1$ is not among the enemies of $a_2$, and reciprocally $a_2$ is not among the enemies of $a_1$.

3) Neither $a_1$ nor $a_2$ is an enemy of itself.

Since $anti(a_i)$ are in general subsets (i.e. the element $a_i$ has one or more enemies), we used braces: writing $\{anti(a_i)\}$.

## *VII.1.20. Definition of n≥2 Neutrosophic Friends*

As an extension of the previous definition, let *(N, *)* be a neutrosophic triplet (strong or weak) set, such that:

$< a_1, neut(a_1), anti(a_1) >$,

$< a_2, neut(a_2), anti(a_2) >$ ,

...,

$< a_n, neut(a_n), anti(a_n) >$,

for $n \geq 2$, be $n$ neutrosophic triplets.

We say that $a_1, a_2, \ldots, a_n$ are *neutrosophic friends* and we write $\mathbb{F} = \{a^1, a^2, \ldots, a_n\}$, if they all have common enemies (denoted by $\mathbb{E}$)

$$\mathbb{E} = \bigcap_{i=1}^{n} \{anti(a_i)\} \neq \emptyset,$$

and none of them is an enemy of another, or an enemy of itself:

$a_i \notin \{anti(a_j)\}$, for any $i, j \in \{1, 2, ..., n\}$.

{ This last relation comprises both: none of them is enemy of the other ( for $i \neq j$ ), and none of them is an enemy of itself ( for $i = j$). }

## *VII.1.21. Proposition*

A neutrosophic enemy of itself has no neutrosophic friend.

## *VII.1.22. Example of Neutrosophic Friends*

If we consider the previous example, $\mathbb{Z}_5 = \{0, 1, 2, 3, 4\}$, endowed with the law $*$ defined by $a * b = 2a + 2b \pmod 5$, for any $a, b \in \mathbb{Z}_5$, whose neutrosophic triplets are:

<0,0,0>, <1,2,0>, <2,4,0>, <3,1,0>, and <4,3,0>, then $\mathbb{F} = \{1, 2, 3, 4\}$ are friends, since they have the same enemy: $anti(1) = anti(2) = anti(3) = anti(4) = 0$, while 0 was excluded, since 0 is an enemy to itself: $anti(0) = 0$.

## *VII.2. Neutrosophic Triplet Function*

*Question from Hur Kul:*

Recently, we had to select only one president from many candidatures.

Then at present about 30 % of the total electors is **movable electors**, i.e., **neutrals**.

Thus, it is very important for them to select whom. But we think that <A>, <neut A>, <anti A> can select partially another candidate, respectively at voting date. So, the final selection is dependent on: <A>, <neut A>, <anti A>. Of course, it is strong dependent to <neut A>.

Hence, we would like to consider (<A>, <neut A>, <anti A>), <f(<A>, <neut A>, <anti A>) in order to analyze the real world.

Your opinion?

*Answer:*

We can define a neutrosophic triplet function:

$$f(<A>,<neutA>,<antiA>) =$$

$$= (f_1(<A>), f_2(<neutA>), f_3(<antiA>)),$$

alike a classical vector function of three variables.

# *VII.3. Theorems on Neutrosophic Triplets*[*]

Firstly, let's recall three definitions that will be used in the next theorems.

## *VII.3.1 Definition 1*

Let (G, *) be a groupoid. An element $a \in G$ is called *left-cancellative* (or has the *left cancellation property*) if for any $b$ and $c$ in $M$, from $a * b = a * c$ one always gets that $b = c$.

## *VII.3.2 Definition 2*

And $a \in G$ is called *right-cancellative* (or has the *right cancellation property*) if for any $b$ and $c$ in $M$, from $b * a = c * a$ one always gets that $b = c$.

## *VII.3.3 Definition 3*

Also, $a \in G$ is called *cancellative* (or has the *two-sided cancellation property*) if $a$ is both left-cancellative and right-cancellative.

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[*] In collab. with Mumtaz Ali.

## *VII.3.4 Definition 4*

A *groupoid* $(G, *)$ is *left-cancellative* (has the left-cancellation property) if all $a \in G$ are left-cancellative;

and similar definitions for the right-cancellative, or two-sided cancellative.

## *VII.3.5. Theorem 1*

*{Improvement of Theorem 3.6 from [1]}*

Let $\mathbb{Z}_p = \{0,1,2,\ldots,p-1\}$, where $p$ is a positive prime number, endowed with the multiplication $\times$ of integers, modulo $p$, with classical unitary element 1.

There exists only one trivial neutrosophic perfect triplet $<0,0,0>$, and $p-1$ neutrosophic imperfect triplets: $<0,0,1>, <0,0,2>, \ldots, <0,0,p-1>$.

$(\mathbb{Z}_p, \times)$, modulo $p$, is a neutrosophic triplet weak set, and it is not a neutrosophic triplet group.

*Proof*

Let's show that $<0,0,i>$, for $i \in \{0,1,2,\ldots,p-1\}$, are neutrosophic triplets.

$0 \times neut(0) = neut(0) \times 0 = 0$, means that

$0 \times 0 = 0 \times 0 = 0$, therefore $neut(0) = 0$;

and

$0 \times anti(0) = anti(0) \times 0 = neut(0)$ means that

$0 \times i = i \times 0 = 0$ or $0 = 0$, therefore $anti(0) = i$.

Let's prove that there are no other neutrosophic triplets.

Suppose there is $a \neq 0$, or $a \in \{1,2,3,\ldots,p-1\}$ such that

$< a, neut(a), anti(a) >$

is a neutrosophic triplet, where $neut(a) \in \mathbb{Z}_p \setminus \{1\}$ and $anti(a) \in \mathbb{Z}_p$.

Let's denote $neut(a) = x$, that we need to find out, when $x \neq 1$ and $x \in \mathbb{Z}_p$.

Since the multiplication is commutative, we need to only check:

$a \times neut(a) = a$,

or $ax = a \pmod{p}$,

or $a(x-1) = 0 \pmod{p}$.

Since $a \neq 0$ and $a < p$, one has $(a,p) = 1$, i.e. $a$ and $p$ are relatively prime.

There one needs $x - 1 = 0 \pmod{p}$, meaning that $x = 1, p+1, 2p+1, 3p+1, \ldots$.

Yet $x \neq 1$ (not allowed to be equal to the classical unitary element) and $x < p$ since $x \in \mathbb{Z}_p$. Therefore, these is no such $x$, i.e. there is no $neut(a)$ for all $a \neq 0$.

In consequence, there is no $anti(a)$ either, for all $a \neq 0$.

## *VII.3.6. Theorem 2*

*{Improvement of Theorem 2.5 from [1]}*

Let $(N,*)$ be a neutrosophic triplet (strong) group.

Let $< a, neut(a), anti(a) >$ be a neutrosophic triplet from $N$.

If $a$ is left- and right-cancellable, then

$< anti(a), neut(a), a >$

and

$< neut(a), neut(a), neut(a) >$

are also neutrosophic triplets.

*Proof*

In order to show that $< anti(a), neut(a), a >$ is a neutrosophic triplet, we only need to show that $neut(a)$ is a neutral for $anti(a)$ too:

$neut(a) * anti(a) = anti(a)$ becomes

$[neut(a) * anti(a)] = anti(a)$

$a * [neut(a) * anti(a)] = a * anti(a)$,

by applying « $a$ » to the left, since $a$ is left-cancellable (so $a \neq 0$, and $a$ is not a zero-divisor),

or $[a * neut(a)] * anti(a) = a * anti(a)$ (associativity),

or $a * anti(a) = a * anti(a)$, which is true;

we used $a * neut(a) = a$ because

$< a, neut(a), anti(a) >$ is a neutrosophic triplet.

Let's also prove that:

$anti(a) * neut(a) = anti(a)$, which becomes

$[anti(a) * neut(a)] = anti(a)$

$[anti(a) * neut(a)] * a = anti(a) * a$

by applying « $a$ » to the right, since $a$ is right-cancellable (so $a \neq 0$, and $a$ is not a zero-divisor),

or *anti(a)*[neut(a)*a] = anti(a)*a* {associativity}

or *anti(a) * a = anti(a) * a* that is true.

Since $a$ is right-cancellable, there was no risk of altering the equality by applying "$a$" to the right in both sides.

Similarly, to show that

$< neut(a), neut(a), neut(a) >$

is a neutrosophic triplet, we only need to show that:

$neut(a) * neut(a) = neut(a)$.

Apply "$a$" to the left, since $a$ is left-cancellable (so in particular $a \neq 0$ and $a$ is not a zero-divisor), therefore there is no risk to alter the equality:

$a * [neut(a) * neut(a)] = a * neut(a)$,

or

$[a * neut(a)] * neut(a) = a * neut(a)$

(associativity),

$a * neut(a) = a * neut(a)$,

or $a = a$ that is true.

## *VII.3.7. Counter-Example 1*

In a neutrosophic triplet group $(NTG, *)$, where $a$ is not left-cancellable (for example $a$ is zero, or $a$ is a zero-divisor, etc.), if $(a, neut(a), anti(a))$ is a neutrosophic triplet, then it may arise that $(anti(a), neut(a), a)$ or *(neut(a), neut(a), neut(a))* are neutrosophic triplets in some cases, and in other cases they may not be neutrosophic triplets.

Let $\mathbb{Z}_{10} = \{0, 1, 2, \ldots, 9\}$, with the integer multiplication modulo *10*, which is a neutrosophic triplet commutative weak set, whose classical unit element is 1.

Then $(2,6,8)$, where 2 is not left-cancellable, is a neutrosophic triplet, and $(8,6,2), (6,6,6)$ are also neutrosophic triplets.

Now, $(2,6,3)$, where 2 is not left-cancellable, is a neutrosophic triplet, however $(3,6,2)$ is not a neutrosophic triplet, because $3 * 6 = 6 * 3 = 8 \neq 3$, while $(6,6,6)$ is a neutrosophic triplet.

Analogously, $(0,0,i)$, where $a = 0$ and $neut(a) = 0$, $anti(a) = i$, for $i \in \{1,2,\ldots,9\}$, are neutrosophic triplets, while <i, 0, 0> are not neutrosophic triplets since $i * 0 = 0 * i \neq i$, while $(0,0,0)$ is a neutrosophic triplet.

## *VII.3.8. Theorem 3*

*{Improvement of Theorem 3.21 from [1]}*

In a neutrosophic triplet group $(NTG,*)$, where $a$ is left-concellative or right-concellative, one has:

$$neut(a) * neut(a) = neut(a), \quad (1)$$

and, in general,

$$neut(a)^n = neut(a^n), \text{ for } n \geq 1; \quad (2)$$

and also: (3)

$$neut(a) * anti(a) = anti(a) * neut(a) = anti(a),$$

and in general: (4)

$$neut(a) * anti(a)^n = anti(a)^n * neut(a) = anti(a)^n.$$

*Proof*

Multiply each equality to the left or to the right with $a$, which is different from 0 and from zero-divisor since $a$ is left-concellative or respectively right-concellative.

## *VII.3.9 Definition of Neutro-Homomorphism.*

*{Improvement of Definition 4.1 from [1]}*

Let $(N_1, *_1)$ and $(N_2, *_2)$ be two neutrosophic triplet groups. A mapping:

$$f: N_1 \rightarrow N_2$$

is called a *neutro-homomorphism* if:

1) *for any* $a, b \in N_1$, *we have:*

$f(a *_1 b) = f(a) *_2 f(b)$;

2) *if* <a, neut(a), anti(a)> *is a neutrosophic triplet from* $N_1$, *then* $f(neut(a)) = neut(f(a))$ *and* $f(anti(a)) = anti(f(a))$.

In other words, if <a, neut(a), anti(a)> is a neutrosophic triplet from $N_1$, then

*<f(a), f(neut(a)), f(anti(a))> is a neutrosophic triplet from $N_2$ that is equal to <f(a), neut(f(a)), anti(f(a))>.*

## *VII.3.10. Example*

Let $N_1$ be a neutrosophic triplet group with respect to multiplication modulo 6 in $(Z_6,\times)$, where

$$N_1 = \{0,2,4\}$$

and let $N_2$ be another neutrosophic triplet group with respect to multiplication modulo 10 in $(Z_{10},\times)$, where

$$N_2 = \{0,2,4,6,8\}.$$

Let

$$f : N_1 \to N_2$$

be a mapping defined as

$$f(0) = 0,\ f(2) = 4,\ f(4) = 6.$$

Then clearly $f$ is a neutro-homomorphism because conditions (1) and (2) are satisfied easily.

The neutrosophic triplets in $N_1$ are *<0,0,0>*, *<2,4,2>*, and *<4,4,4>*. Then *<f(0),f(0),f(0)> = <0,0,0>* and *<f(2),f(4),f(2)> = <4,6,4>* and

*<f(4),f(4),f(4)> = <6,6,6>* are neutrosophic triplets in $N_2$.

## *VII.3.11. Definition 6*

A neutro-homomorphism is called neutro-isomorphism if it is one-one and onto.

## *VII.3.12. Proposition 1*

Every neutro-homomorphism is a classical homomorphism by neglecting the classical unity element in classical homomorphism.

*Proof.*

First, we neglect the classical unity element that classical homomorphism maps unity element to the corresponding unity element. Now suppose that $f$ is a neutro-homomorphism from a neutrosophic triplet group $N_1$ to a neutrosophic triplet group $N_2$. Then by condition $(1)$, it follows that $f$ is a classical homomorphism.

## *VII.3.13. Proposition 2*

*{Improvement of Proposition 3.11 from [1]}*

Let (N, *) be e neutrosophic triplet group, and let *a, b, c* ∈ *N.*

1) *If a and anti(a) are left-cancellable, then: a*b =a*c if and only if neut(a)*b = neut(a)*c.*

Proof: multiply with *anti(a)* to the left the first equality; conversably multiply by *a* the second equality.

2) *If a and anti(a) are right-concellable, then: b*a = c*a if and only if b*neut(a) = c*neut(a).*

   *Similar proof.*

   *{These (1) and (2) are improvements of Proposition 3.11 from [1].}*

3) *Let a be left-cancellable; if anti(a)*b = anti(a)*c then neut(a)*b = neut(a)*c.*

   *Proof: multiply the first equality with a to the left.*

4) *Let a be right-cancellable; if b*anti(a) = c*anti(b), then b*neut(a) = c*neut(a).*

   *Proof: multiply the first equality by a to the right side.*

*{These (3) and (4) are improvements of Proposition 3.12 from [1].}*

## *VII.3.14. Theorem 4*

*{Combination of Theorems 3.13 & 3.14 from [1]}*

In a neutrosophic triplet commutative group, if

$\langle a, neut(a), anti(a)\rangle$

and

$\langle b, neut(b), anti(b)\rangle$

are two neutrosophic triplets, then

$$\langle a * b, neut(a) * neut(b), anti(a) * anti(b)\rangle$$

is also a neutrosophic triplet;

and the later neutrosophic triplet is equal to

$\langle a * b, neut(a * b), anti(a * b)\rangle$.

*Improved Proofs of 3.13 & 3.14 from [1]:*

$$[a * b] * [neut(a) * neut(b)] =$$

$$= [a * neut(a)] * [b * neut(b)] = a * b, \qquad (5)$$

and

$$[neut(a) * neut(b)] * [a * b] =$$

$$= [neut(a) * a] * [neut(b) * b] = a * b. \qquad (6)$$

That means:

$$neut(a) * neut(b) = neut(a * b). \qquad (7)$$

Similarly, $[a * b] * [anti(a) * anti(b)] =$

$= [a * anti(a)] * [b * anti(b)] =$

$$= neut(a) * neut(b) = neut(a * b), \quad (8)$$

and

$$[anti(a) * anti(b)] * [a * b] =$$
$$= [anti(a) * a] * [anti(b) * b] =$$
$$= neut(a) * neut(b) = neut(a * b), \quad (9)$$

which means:

$$anti(a) * anti(b) = anti(a * b). \quad (10)$$

This can be generalized to the following:

## *VII.3.15. Theorem 5*

In a neutrosophic triplet commutative group, if $\langle a_i, neut(a_i), anti(a_i)\rangle$, for $1 \leq i \leq n$, and $n \geq 2$, are neutrosophic triplets, then:

$$\left\langle \begin{matrix} a_1 * a_2 * \ldots * a_n, neut(a_1) * neut(a_2) * \ldots * neut(a_n), \\ anti(a_1) * anti(a_2) * \ldots * anti(a_n) \end{matrix} \right\rangle$$

is also a neutrosophic triplet, which is equal to:

$$\left\langle \begin{matrix} a_1 * a_2 * \ldots * a_n, neut(a_1 * a_2 * \ldots * a_n), \\ anti(a_1 * a_2 * \ldots * a_n) \end{matrix} \right\rangle.$$

*Proof*

By mathematical induction, using the previous theorem.

*Consequence*

In a neutrosophic triplet commutative group, if $\langle a, neut(a), anti(a)\rangle$ is a neutrosophic triplet,

and $n \geq 2$, then:

$\langle a^n, neut(a^n), anti(a^n) \rangle$

is also a neutrosophic triplet, where

$a^n = \underbrace{a * a * \ldots * a}_{n\ times}$.

*Proof*

In the previous theorem we just set

$a_1 = a_2 = \cdots = a_n$.

## *VII.3.16. Theorem 6*

*{Proposition 3.18 from [1]}*

Let *(N,*)* be a neutrosophic triplet group. A subset *H* of *N* is a neutrosophic triplet subgroup of *N* if and only if:

For any $a, b \in H$, $a*b \in H$;

And for each $a \in H$, the exist $neut(a) \in H$ and $anti(a) \in H$.

## *Reference*

1. Florentin Smarandache, Mumtaz Ali, *Neutrosophic Triplet Group*, Neural Computing and Appl., Springer, 1-7, 2016;
*https://link.springer.com/article/10.1007/s00521-016-2535-x*; DOI: 10.1007/s00521-016-2535-x.

# *VII.4. Neutrosophic Triplet Group vs. Generalized Group.*

The distinctions between Molaei's [7] Generalized Group (GG) and Neutrosophic Triplet Group (NTG) is that in NTG for each $x$ there may exist more neut($x$)'s and/or more anti($x$)'s, while in the GG for each $x$ there is only one neutral and only one inverse for each $x$.

Another distinction is that a commutative GG is a commutative classical group [i.e. the commutative GG has the same neutral for all of its elements – as in the classical group], making the GG less interesting, while a commutative NTG is not reduced to a classical group.

For example:

The neutrosophic triplet strong set $(N, *)$,

$N = \{a, b\}$ defined by:

| $*$ | $\boldsymbol{a}$ | $\boldsymbol{b}$ |
|---|---|---|
| $\boldsymbol{a}$ | $a$ | $b$ |
| $\boldsymbol{b}$ | $b$ | $b$ |

is a commutative neutrosophic triplet strong group, with neutrosophic triplets *<a,a,a>* and *<b,b,b>,* but their neutrals are different:

$$neut(a) = a \neq b = neut(b),$$

therefore *(N, *)* is not a classical group: since it does not have a unitary element, nor inverse elements.

Similarity between the non-commutative GG and the NTG is that the neutral is different from an element to another, unlike in the classical group where there is a single neutral, the same, for all elements *x* into the classical group.

## *VII.4.1. Example*

Below, an example of Neutrosophic Triplet Strong Set (not necessarily group, since the law * is non-associative). Let the set L = {a, b, c, d}, endowed with the law * defined according to the Cayley Table below:

| * | ***a*** | ***b*** | ***c*** | ***d*** |
|---|---|---|---|---|
| ***a*** | *a* | *a* | *b* | *b* |
| ***b*** | *a* | *d* | *b* | *c* |

| | | | | |
|---|---|---|---|---|
| ***c*** | *b* | *b* | *c* | *d* |
| ***d*** | *b* | *c* | *d* | *c* |

with the following neutrosophic triplets:

<a, a, a>, <a, b, c>, <a, b, d> {therefore "a" has 2 neutrals: a, b; and 3 anti's: a, c, d};

<b, c, d>;

<c, c, c>;

<d, c, b>, <d, c, d> {therefore "d" has 2 anti's}.

## *VII.4.2. Example*

Another example, of Neutrosophic triplet Weak Set, where the law is associative and commutative, but an element *x* has many anti(*x*)'s.

In $Z_{10}$ = {0, 1, 2, ..., 9}, with the classical multiplication modulo 10 (*), one has:

<0, 0, 0>, <0, 0, 1>, <0, 0, 2>, ..., <0, 0, 9>

so for 0 one has:

neut(0) = 0, but ten anti(0)'s = 0, 1, 2, ..., 9;

<2, 6, 3>, <2, 6, 8>; so two anti(2)'s = 3, 8;

<4, 6, 4>, <4, 6, 9>; so two anti(4)'s = 4, 9;

<5, 5, 1>, <5, 5, 3>, <5, 5, 5>, <5, 5, 7>, <5, 5, 9>, so four anti(5)'s = 1, 3, 5, 7, 9;

<6, 6, 1>, <6, 6, 6>, so two anti(6)'s = 1, 6;

<8, 6, 2>, <8, 6, 7>, so two anti(8)'s = 2, 7.

## *Reference:*

[1] Molaei M.R. (1999). *Generalized groups*. Bul. Inst. Politehn. Iasi, Sect I 45(49): 21–24.

# *VII.5. Neutrosophic Triplet Multiple Order*[*]

In general, an element $a$ may have many *neut(a)*'s. So, when one defines the neutrosophic triplet order of $a$, denoted as *nto(a)*, this is defined with respect to a specific *neut(a)*.

Therefore, let's say that $neut(a) = \{b_1, b_2\}$.

Then, the neutrosophic triplet order of $a$ with respect to $neut(a) = b_1$ may be $n_1$, which means that $n_1$ is the smallest positive integer $\geq 1$ such that

$a^{n1} = b_1$;

while the neutrosophic triplet order of $a$ with respect to $neut(a) = b_2$ may be $n_2$, which means that $n_2$ is the smallest positive integer $\geq 1$ such that

$a^{n2} = b_2$;

with $n_1$ in general different from $n_2$.

This definition is an improvement of the *Definition 3.19*, from [1].

[*] An answer to J. Kim, K. Hur, P.K. Lim, J.G. Lee. In collaboration with Mumtaz Ali.

## *Reference:*

[1] Florentin Smarandache, Mumtaz Ali, *Neutrosophic Triplet Group*, Neural Computing and Appl., Springer, 1-7, 2016;
https://link.springer.com/article/10.1007/s00521-016-2535-x; DOI: 10.1007/s00521-016-2535-x.

**CHAPTER VIII**

# *VIII.1. Neutrosophic Triplet Ring*[*]

## *VIII.1.1. Definition of Neutrosophic Triplet Ring*

The Neutrosophic Triplet Ring (*NTR*) is a set endowed with two binary laws *(M, *, #)*, such that:

a) *(M, *)* is a commutative neutrosophic triplet group; which means that:

- *M* is a strong set of neutrosophic triplets with respect to the law * (i.e. if *x* belongs to *M*, then *neut(x)* and *anti(x),* defined with respect to the law *, also belong to *M*);

- the law $*$ is well-defined, associative, and commutative on *M* (as in the classical sense);

b) *(M, #)* is a set such that the law # on *M* is well-defined and associative (as in the classical sense);

c) the law # is distributive with respect to the law $*$ (as in the classical sense).

[*] In collaboration with Mumtaz Ali.

*Remarks*

a) The Neutrosophic Triplet Ring is defined on the steps of the classical ring, the only two distinctions are that:

- the classical unit element with respect to the law $*$ is replaced by *neut(x)* with respect to the law $*$ for each $x$ in $M$ into the *NTR*;

- in the same way, the classical inverse element of an element $x$ in $M$, with respect to the law $*$, is replaced by *anti(x)* with respect to the law $*$ in $M$.

b) A Neutrosophic Triplet Ring, in general, is different from a classical ring.

## *VIII.2. Hybrid Neutrosophic Triplet Ring*

### *VIII.2.1. Definition*

The Hybrid Neutrosophic Triplet Ring of First Type (*HNTR1*) is a set endowed with two binary laws $(M,*,\#)$, such that:

a) $(M,*)$ is a commutative neutrosophic triplet group; which means that:

- $M$ is a strong set of neutrosophic triplets with respect to the law $*$ (i.e. if $x$ belongs to $M$, then *neut(x)* and *anti(x)*, defined with respect to the law $*$, also belong to $M$);

- the law $*$ is well-defined, associative, and commutative on $M$ (as in the classical sense);

b) $(M,\#)$ is a neutrosophic triplet strong set with respect to the law # (i.e. if $x$ belongs to $M$, then *neut(x)* and *anti(x)*, defined with respect to the law #, also belong to $M$);

- the law # is well-defined and non-associative on $M$ (as in the classical sense);

c) the law # is distributive with respect to the law $*$ (as in the classical sense).

*Remarks*

a) A Hybrid Neutrosophic Triplet Ring of First Type (*HNTR1*) is a neutrosophic triplet field $(F,*,\#)$ from which there has been removed the associativity of the second law #.

b) Or, Hybrid Neutrosophic Triplet Ring of First (*HNTR1*) is a set $(F,*,\#)$, such that $(F,*)$ is a commutative neutrosophic triplet group, and $(F,\#)$ is a neutrosophic triplet loop, and the law # is distributive with respect to the law $*$ (as in the classical sense).

# *VIII.3. Hybrid Neutrosophic Triplet Ring of Second Type*

## *VIII.3.1. Definition*

The Hybrid Neutrosophic Triplet Ring of Second Type (*HNTR2*) is a set endowed with two binary laws $(M,*,\#)$, such that:

a) $(M,*)$ is a commutative neutrosophic triplet group; which means that:

- $M$ is a strong set of neutrosophic triplets with respect to the law $*$ (i.e. if $x$ belongs to $M$, then *neut(x)* and *anti(x)*, defined with respect to the law $*$, also belong to $M$);

- the law $*$ is well-defined, associative, and commutative on $M$ (as in the classical sense);

b) $(M,\#)$ is a neutrosophic triplet weak set with respect to the law # { i.e. if $x$ belongs to $M$, then there exist a neutrosophic triplet in M with respect to the law #, <y, *neut(y)* and *anti(y)*, such that $x = y$ or $x =$ *neut(y)* or $x =$ *anti(y)* };

- the law # is well-defined and associative on $M$ (as in the classical sense);

c) the law # is distributive with respect to the law $*$ (as in the classical sense).

*Remarks*

a) A Hybrid Neutrosophic Triplet Ring of Second Type (*HNTR2*) is a neutrosophic triplet field $(F,*,\#)$ from which there has been removed the existence of neutrals and opposites with respect to the second law #.

b) Or, Hybrid Neutrosophic Triplet Ring of Second Type (*HNTR2*) is a set $(F,*,\#)$, such that $(F,*)$ is a commutative neutrosophic triplet stromg group, and $(F,\#)$ is a neutrosophic triplet weak group, and the law # is distributive with respect to the law $*$ (as in the classical sense).

# *VIII.4. Neutrosophic Triplet Field*[*]

## *VIII.4.1. Definition*

Neutrosophic Triplet Field (*NTF*) is a set endowed with two binary laws *(M, *, #),* such that:

a) *(M, *)* is a commutative neutrosophic triplet group; which means that:

- M is a strong set of neutrosophic triplets with respect to the law * (i.e. if *x* belongs to *M*, then *neut(x)* and *anti(x),* defined with respect to the law *, also both belong to *M*);

- the law * is well-defined, associative, and commutative on *M* (as in the classical sense);

b) *(M, #)* is a neutrosophic triplet strong group; which means that:

- *M* is a strong set of neutrosophic triplets with respect to the law # (i.e. if *x* belongs to *M*, then *neut(x)* and *anti(x),* defined with respect to the law #, also both belong to *M*);

- the law # is well-defined and associative on *M* (as in the classical sense);

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[*] In collaboration with Mumtaz Ali.

c) the law # is distributive with respect to the law * (as in the classical sense).

*Remarks*

1) The Neutrosophic Triplet Field is defined on the steps of the classical field, the only four distinctions are that:

- the classical unit element with respect to the first law * is replaced by *neut(x)* with respect to the first law * for each *x* in *M* into the *NTF*;

- in the same way, the classical inverse element of an element *x* in *M*, with respect to the first law *, is replaced by *anti(x)* with respect to the first law * in *M*;

- and the classical unit element with respect to the second law # is replaced by *neut(x)* with respect to the second law # for each *x* in *M* into the *NTF*;

- in the same way, the classical inverse element of an element *x* in *M*, with respect to the second law #, is replaced by *anti(x)* with respect to the second law # in *M*;

2) A Neutrosophic Triplet Field, in general, is different from a classical field.

VIII.4.2. *Example of Neutrosophic Triplet Ring which is not a Neutrosophic Triplet Field.*

Let $(N,*) = \{a, b, c\}$, defined as in the table below:

| $*$ | $a$ | $b$ | $c$ |
|---|---|---|---|
| $a$ | b | c | a |
| $b$ | c | a | b |
| $c$ | a | b | c |

Neutrosophic Triplets are:

(a, c, b) since ac = ca = a

and ab = ba = b

also bc = cb = b

(b, c, a) since bc = cb = b

and ab = ba = c

also ac = ca = c

(c, c, c) since cc = c

Let $(N,\#) = \{a, b, c\}$, defined as in the table below:

| $\#$ | $a$ | $b$ | $c$ |
|---|---|---|---|
| $a$ | c | a | c |
| $b$ | a | ⓐ | b |
| $c$ | c | b | c |

$(a, b, \nexists)$,

$(b, c, \nexists)$,

$(c, a, \nexists)$.

For $a \in (N, \#)$, there is $neut(a) = b$, but there is no $anti(a)$.

For $b \in (N, \#)$, there is $neut(b) = c$, but there is no $anti(b)$.

For $c \in (N, \#)$, there is $neut(c) = a$, but there is no $anti(c)$.

Hence $(N, *, \#)$ is a neutrosophic triplet ring, but it is not a neutrosophic triplet field.

# *VIII.5. Hybrid Neutrosophic Triplet Field*[*]

## *VIII.5.1. Hybrid Neutrosophic Triplet Field of Type 1.*

It is a set *F* endowed with two laws * and # such that:

1: *(F, *)* is a commutative neutrosophic triplet strong group;

2: *(F, #)* is a classical group;

3: The law # is distributive over the law *.

## *VIII.5.2. Hybrid Neutrosophic Triplet Field of Type 2.*

It is a set *F* endowed with two laws * and # such that:

1: *(F, *)* is a classical commutative group;

2: *(F, #)* is a neutrosophic triplet strong group;

3: The law # is distributive over the law *.

[*] In collaboration with Mumtaz Ali.

## *VIII.6. Neutrosophic Triplet Loop*

We define the Neutrosophic Triplet Loop in the following way:

A set $(L,*)$ such that:

1) the law * is well defined;

2) for each element $a$ in $L$, there exists a *neut(a)* in $L$, such that:

$$a * neut(a) = neut(a) * a = a; \quad (1)$$

3) for each element $a$ in $L$, there exists an *anti(a)* in $L$, such that:

$$a * anti(a) = anti(a) * a = neut(a). \quad (2)$$

{The law * may be non-associative.}

Let's see an example.

In $(\mathbb{Z}_{10}, *)$, the set of integers modulo $10$, where for any $x, y$ in $\mathbb{Z}_{10}$, $x * y = 2x + 2y \ (mod\ 10)$.

The law is non-associative, since:

$$(x * y) * z = x * (y * z) \quad (3)$$

produces:

$$(2x + 2y) * z = x * (2y + 2z), \quad (4)$$

or

$$4x + 4y + 2z = 2x + 4y + 4z, \quad (5)$$

which in general is false.

The law * has no unit element $e$ in the classical sense, since

$$x * e = 2x + 2e = x, \tag{6}$$

or

$$2e = -x, \tag{7}$$

so $e$ depends on $x$, which doesn't work.

One finds the following neutrosophic triplets:

*(0, 0, 0), (0, 0, 5), (2, 4, 0), (2, 4, 5), (4, 8, 0), (4, 8, 5), (6, 2, 0), (6, 2, 5), (8, 6, 0), (8, 6, 5).*

Thus, the set:

$L = \{0, 2, 4, 5, 6, 8\}$,

with the non-associative law

$$x * y = 2x + 2y, \tag{8}$$

is a neutrosophic triplet loop.

## *VIII.7. Neutrosophic Triplet Structures*

The neutrosophic triplets and their algebraic structures were first time introduced by Florentin Smarandache and Mumtaz Ali in 2014 - 2016 [1,2].

They are derived from neutrosophy [4], founded in *1995*, which is a generalization of dialectics, and it is a new branch of philosophy that studies the origin, nature, and scope of neutralities, as well as their interactions with different ideational spectra. Neutrosophy is also the basis of neutrosophic logic, neutrosophic probability, neutrosophic set, neutrosophic statistics, neutrosophic algebraic structures and so on. Neutrosophy and its neutrosophic derivatives are based on triads of the form *(<A>, <neutA>, <antiA>),* where *<A>* is an entity, *<antiA>* is the opposite of *<A>,* while *<neutA>* is the neutral between *<A>* and *<antiA>.*

The set of neutrosophic triplets, embedded with a well-defined law * that satisfies some axioms, form the neutrosophic triplet algebraic structures.

The first studied structure was the neutrosophic triplet group [1]. The neutrosophic triplet algebraic structures follow on the steps of classical algebraic structures, with two distinctions:

- the classical unit element is replaced by the neutrosophic neut(a)'s;

- and the classical inverse element is replaced by the neutrosophic anti(a)'s.

## *References:*

[1] Florentin Smarandache, Mumtaz Ali, *Neutrosophic Triplet Group*, Neural Computing and Appl., Springer, 1-7, 2016, *https://link.springer.com/article/10.1007/s00521-016-2535-x*; DOI: 10.1007/s00521-016-2535-x.

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*http://meetings.aps.org/Meeting/GEC16/Session/HT6.111.*

[3] Florentin Smarandache, Mumtaz Ali, *The Neutrosophic Triplet Group and its Application to Physics*, Universidad Nacional de Quilmes, Depart. of Science & Technology, Buenos Aires, 02 June 2014.

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[5] F. Smarandache, *Neutrosophic Perspectives: Triplets, Duplets, Multisets, Hybrid Operators, Modal Logic, Hedge Algebras. And Applications*, 1st ed., Pons Editions, Bruxelles, 324 p., 2017.

[6] F. Smarandache, Neutrosophic Triplets, University of New Mexico, USA, *http://fs.gallup.unm.edu/NeutrosophicTriplets.htm*

[7] Molaei M.R. (1999). *Generalized groups*. Bul. Inst. Politehn. Iasi, Sect I 45(49): 21–24.

**CHAPTER IX**

# *IX.1. Neutrosophic Duplets*

The Neutrosophic Duplets and their algebraic structures were first introduced by the author in [1].

## *IX.1.1. Definition of Neutrosophic Duplet.*

Let $\mathcal{U}$ be a universe of discourse, and a set $A \subset \mathcal{U}$, endowed with a well-defined law $*$.

We say that $< a, neut(a) >$, where $a, neut(a) \in A$ is a *neutrosophic duplet* if:

1) $neut(a)$ is different from the unit element of $A$ with respect to the law * (if any);

2) $a * neut(a) = neut(a) * a = a$;

3) there is no $anti(a) \in A$ such that

$a * anti(a) = anti(a) * a = neut(a)$.

## *IX.1.2. Example of Neutrosophic Duplets.*

In $(\mathbb{Z}_8, *)$, the set of integers modulo 8; with respect to the regular multiplication $*$ one has the following neutrosophic duplets:

$< 2, 5 >, < 4, 3 >, < 4, 5 >, < 4, 7 >, < 6, 5 >$.

*Proof*

$\mathbb{Z}_8 = \{0, 1, 2, 3, \ldots, 7\}$, with unitary element *1* for the multiplication * modulo *8*.

$2 \times 5 = 5 \times 2 = 10 = 2$ *(mod 8);*

so $neut(2) = 5 \neq 1$.

There is no *anti(2)* $\in \mathbb{Z}_8$, because:

$2 \times anti(2) = 5$ *(mod 8),*

or $2y = 5$ *(mod 8)* by denoting $anti(2) = y$, is equivalent to:

$2y - 5 = \mathcal{M}_8$ {*multiple of 8*}, or $2y - 5 = 8k$, where $k$ is an integer, or

$2(y - 4k) = 5$, where both $y, k$ are integers, or:

*even number = odd number,*

which is impossible.

Therefore, we proved that *<2, 5>* is a neutrosophic duplet.

Similarly for *<4, 5>, <4, 3>, <4, 7>, <6, 5>.*

A counter-example: *<0, 0>* is not a neutrosophic duplet, because it is a neutrosophic triplet:

<0, 0, 0>, where there exists an $anti(0) = 0$.

## *Reference*

[1] F. Smarandache, *Neutrosophic Duplet Structures*, Meeting of the Texas Section of the APS, Texas Section of the AAPT, The University of Texas at Dallas, Richardson, Texas, 2017.

# *IX.2. Neutrosophic Duplet Set and Neutrosophic Duplet Structures*

## *IX.2.1. Definition of Neutrosophic Duplet Strong Set*

A *Neutrosophic Duplet Strong Set*, $(D,*)$, is a set $D$, endowed with aa well-defined binary law $*$, such that $\forall a \in D$, $\exists$ *neut(a)* $\in D$.

Therefore, any element $a$ from $D$ forms a neutrosophic duplet *<a, neut(a)>* in *D.*

## *IX.2.2. Definition of Neutrosophic Duplet Weak Set*

A *Neutrosophic Duplet Weak Set*, $(D,*)$, is a set $D$, endowed with a well-defined binary law $*$, such that $\forall a \in D$, there exist a neutrosophic duplet $\langle b, neut(b)\rangle$ such that $\langle b, neut(b)\rangle \subseteq D$ and $a = b$ or $a = neut(b)$.

Therefore, any element from $D$ belongs to at least a neutrosophic duplet.

### *IX.2.3. Proposition.*

Any neutrosophic duplet strong set is also a neutrosophic duplet weak set, but not conversely.

### *IX.2.4. Theorem*

The richest possible structure is the Neutrosophic Duplet Commutative Strong Semigroup with Neutrosophic Neutrals, i.e.

(1) $\forall a, b \in D, a * b \in D$;

(2) $\forall a, b, c \in D, a * (b * c) = (a * b) * c$;

(3) $\forall a, b \in D, a * b = b * a$;

(4) $\forall a \in D$, $\exists\, neut(a) \in D$, such that $a * neut(a) = neut(a) * a = a$;

(5) $\forall a \in D$, $\nexists\, anti(a) \in D$, such that $a * anti(a) = anti(a) * a = a$.

In other words, the *Neutrosophic Duplet Strong Set* can be defined as follows:

— for any *x* in *D*, there is a *neut(x)* in *D*, such that

$x * neut(x) = neut(x) * x = x$, (1)

— and there is no *anti(x)* in *D* for which

$x * anti(x) = neut(x)$ or $anti(x) * x = x$. (2)

## *IX.2.5. Example of Neutrosophic Duplet Strong Set*

Let *ND = {a, b, c}* be a set endowed with the law * as defined in the below Cayley Table:

| * | ***a*** | ***b*** | ***c*** |
|---|---|---|---|
| *a* | *c* | *a* | *c* |
| *b* | *a* | *a* | *b* |
| *c* | *c* | *b* | *b* |

The law * is well-defined (according to the above table), commutative, since the table's matrix

| | | |
|---|---|---|
| *c* | *a* | *c* |
| *a* | *a* | *b* |
| *c* | *b* | *b* |

is symmetric with respect to its main diagonal, but it is not associative since, for example:

$$(a*b)*c = a*(b*c) \qquad (3)$$

produces $a*c = a*b$ or $c = a$, which is false.

The neutrosophic duplets are: *<a, b>*, *<b, c>*, and *<c, a>*.

For each element $x$ in $ND$ there exists a neutrosophic neutral and no neutrosophic inverse:

$neut(a) = b$, and $anti(a)$ does not exist since
$a*x \neq b$ for any $x \in ND$;
$neut(b) = c$, and $anti(b)$ does not exist since
$b*x \neq c$ for any $x \in ND$;
$neut(c) = a$, and $anti(c)$ does not exist since
$c*x \neq a$ for any $x \in ND$.

The neutrosophic duplets have the general form $<x, neut(x), no\ anti(x)>$ with respect to the neutrosophic triplet form and with neutrosophy, and as applications of neutrosophic duplets there are items $x$ that have no opposites. For example, several species of animals and plants in Galapagos Archipelago that have no predators.

Further: we can develop a new type of structures: *Neutrosophic Duplet Structures*, which are structures defined on the strong or weak sets of neutrosophic duplets.

# References

1. F. Smarandache, *Neutrosophic Duplet Structures*, Meeting of the Texas Section of the APS, Texas Section of the AAPT, and Zone 13 of the Society of Physics Students, The University of Texas at Dallas, Richardson, Texas, 2017.

2. F. Smarandache, *Neutrosophic Duplets*, University of New Mexico, Gallup Campus, USA; *http://fs.gallup.unm.edu/NeutrosophicDuplets.htm*

**CHAPTER X**

# *X.1. Neutrosophic Multiset*

Let $\mathcal{U}$ be a universe of discourse, and $M \subset \mathcal{U}$.

## *X.1.1. Definition*

A *Neutrosophic Multiset M* is a neutrosophic set where one or more elements are repeated with the same neutrosophic components, or with different neutrosophic components.

## *X.1.2. Examples*

$A = \{a(0.6, 0.3, 0.1), b(0.8, 0.4, 0.2), c(0.5, 0.1, 0.3)\}$

is a neutrosophic set (not multiset).

But

$B = \{a(0.6, 0.3, 0.1), a(0.6, 0.3, 0.1), b(0.8, 0.4, 0.2)\}$

is a neutrosophic multiset, since the element $a$ is repeated; we say that the element $a$ has *neutrosophic multiplicity 2* with the same neutrosophic components.

While

$$C = \begin{Bmatrix} a(0.6, 0.3, 0.1), a(0.7, 0.1, 0.2), \\ a(0.5, 0.4, 0.3), c(0.5, 0.1, 0.3) \end{Bmatrix}$$

is also a neutrosophic multiset, since the element $a$ is repeated (it has *neutrosophic multiplicity 3*),

but with different neutrosophic components, since, for example, during the time, the neutrosophic membership of an element may change.

If the element $a$ is repeated $k$ times keeping the same neutrosophic components $(t_a, i_a, f_a)$, we say that $a$ has *multiplicity* $k$.

But if there is some change in the neutrosophic components of $a$, we say that $a$ has the *neutrosophic multiplicity* $k$.

Therefore, we define in general the *Neutrosophic Multiplicity Function*:

$$nm: \mathcal{U} \to \mathbb{N},$$

where $\mathbb{N} = \{1, 2, 3, \dots, \infty\}$,

and for any $a \in A$ one has (1)

$$nm(a) = \{(k_1, \langle t_1, i_1, f_1 \rangle), (k_2, \langle t_2, i_2, f_2 \rangle), \dots, (k_j, \langle t_j, i_j, f_j \rangle), \dots\}$$

which means that $a$ is repeated $k_1$ times with the neutrosophic components $\langle t_1, i_1, f_1 \rangle$; $a$ is repeated $k_2$ times with the neutrosophic components $\langle t_2, i_2, f_2 \rangle$, ..., $a$ is repeated $k_j$ times with the neutrosophic components $\langle t_j, i_j, f_j \rangle$, ..., and so on.

Of course, all $k_1, k_2, \dots, k_j, \dots \in \mathbb{N}$, and $\langle t_p, i_p, f_p \rangle \neq \langle t_r, i_r, f_r \rangle$, for $p \neq r$, with $p, r \in \mathbb{N}$.

Also, all neutrosophic components are with respect to the set $A$. Then, a neutrosophic multiset $A$ can be written as:

$$\big(A, nm(a)\big)$$

or $\{(a, nm(a), \text{for } a \in A)\}$.

## *X.1.3. Examples of operations with neutrosophic multisets.*

Let's have:

$$A = \{5_{\langle 0.6,0.3,0.2\rangle}, 5_{\langle 0.6,0.3,0.2\rangle}, 5_{\langle 0.4,0.1,0.3\rangle}, 6_{\langle 0.2,0.7,0.0\rangle}\}$$

$$B = \{5_{\langle 0.6,0.3,0.2\rangle}, 5_{\langle 0.8,0.1,0.1\rangle}, 6_{\langle 0.9,0.0,0.0\rangle}\}$$

$$C = \{5_{\langle 0.6,0.3,0.2\rangle}, 5_{\langle 0.6,0.3,0.2\rangle}\}.$$

Then:

### *X.1.3.1. Intersection of Neutrosophic Multisets.*

$$A \cap B = \{5_{\langle 0.6,0.3,0.2\rangle}\}.$$

### *X.1.3.2. Union of Neutrosophic Multisets.*

$$A \cup B = \left\{ \begin{matrix} 5_{\langle 0.6,0.3,0.2\rangle}, 5_{\langle 0.6,0.3,0.2\rangle}, 5_{\langle 0.4,0.1,0.3\rangle}, 5_{\langle 0.8,0.1,0.1\rangle}, \\ 6_{\langle 0.2,0.7,0.0\rangle}, 6_{\langle 0.9,0.0,0.0\rangle} \end{matrix} \right\}.$$

### *X.1.3.3. Inclusion of Neutrosophic Multisets.*

$C \subset A$, but $C \not\subset B$

### *X.1.3.4. Cardinality of Neutrosophic Multisets.*

$Card(A) = 4$ , and $Card(B) = 3$, where $Card(\cdot)$ means cardinal.

### *X.1.3.5. Cartesian Product of Neutrosophic Multisets.*

$$B \times C$$

$$= \begin{Bmatrix} (5_{\langle 0.6,0.3,0.2\rangle}, 5_{\langle 0.6,0.3,0.2\rangle}), (5_{\langle 0.6,0.3,0.2\rangle}, 5_{\langle 0.6,0.3,0.2\rangle}), \\ (5_{\langle 0.8,0.1,0.1\rangle}, 5_{\langle 0.6,0.3,0.2\rangle}), (5_{\langle 0.8,0.1,0.1\rangle}, 5_{\langle 0.6,0.3,0.2\rangle}), \\ (6_{\langle 0.9,0.0,0.0\rangle}, 5_{\langle 0.6,0.3,0.2\rangle}), (6_{\langle 0.9,0.0,0.0\rangle}, 5_{\langle 0.6,0.3,0.2\rangle}) \end{Bmatrix}.$$

### *X.1.3.6. Difference of Neutrosophic Multisets.*

$$A - B = \{5_{\langle 0.6,0.3,0.2\rangle}, 5_{\langle 0.4,0.1,0.3\rangle}, 6_{\langle 0.2,0.7,0.0\rangle}\}$$

$$A - C = \{5_{\langle 0.4,0.1,0.3\rangle}, 6_{\langle 0.2,0.7,0.0\rangle}\}$$

$$C - B = \{5_{\langle 0.6,0.3,0.2\rangle}\}$$

### *X.1.3.7. Sum of Neutrosophic Multisets.*

$$A \uplus B$$

$$= \begin{Bmatrix} 5_{\langle 0.6,0.3,0.2\rangle}, 5_{\langle 0.6,0.3,0.2\rangle}, 5_{\langle 0.6,0.3,0.2\rangle}, 5_{\langle 0.4,0.1,0.3\rangle}, 5_{\langle 0.8,0.1,0.1\rangle}, \\ 6_{\langle 0.2,0.7,0.9\rangle}, 6_{\langle 0.9,0.0,0.0\rangle} \end{Bmatrix}$$

$$B \uplus B = \begin{Bmatrix} 5_{\langle 0.6,0.3,0.2\rangle}, 5_{\langle 0.6,0.3,0.2\rangle}, 5_{\langle 0.8,0.1,0.1\rangle}, 5_{\langle 0.8,0.1,0.1\rangle}, \\ 6_{\langle 0.9,0.0,0.0\rangle}, 6_{\langle 0.9,0.0,0.0\rangle} \end{Bmatrix}.$$

Let's compute the neutrosophic multiplicity function, with respect to several of the previous neutrosophic multisets.

$$nm_A: A \to \mathbb{N}$$

$nm_A(5) = \{(2, \langle 0.6, 0.3, 0.2\rangle), (1, \langle 0.4, 0.1, 0.3\rangle)\}$

$nm_A(6) = \{(1, \langle 0.2, 0.7, 0.0\rangle)\}.$

$nm_B: B \to \mathbb{N}$

$nm_B(5) = \{(1, \langle 0.6, 0.3, 0.2\rangle), (1, \langle 0.8, 0.1, 0.1\rangle)\}$

$nB(6) = \{(1, \langle 0.2, 0.7, 0.0\rangle)\}.$

$nm_C: C \to \mathbb{N}$

$nm_C(5) = \{(2, \langle 0.6, 0.3, 0.2\rangle)\}$

## *References*

1. Eric W. Weisstein, *Multiset*, MathWorld, A Wolfram Web Resource.
*http://mathworld.wolfram.com/Multiset.html*

2. F. Smarandache, *Neutrosphic Multiset Applied in Physical Proceses*, Actualization of the Internet of Things, a FIAP Industrial Physics Conference, Monterey, California, Jan. 2017

# *X.2. Neutrosophic Multiset Applied in Physical Processes*

Let $U$ be a universe of discourse and a set $M \subseteq U$. The *Neutrosophic Multiset M* is defined as a neutrosophic set with the property that one or more elements are repeated either with the same neutrosophic components, or with different neutrosophic components.

For example, *Q = {a(0.6,0.3,0.2), a(0.6,0.3,0.2), a(0.5,0.4,0.1), b(0.7,0.1,0.1)}* is a neutrosophic multiset.

The Neutrosophic Multiplicity Function is defined as:

$$nm: U \rightarrow N = \{1, 2, 3, \dots\},$$

and for each x∈M one has

$$nm(x) = \{(k_1, < t_1, i_1, f_1 >,),$$
$$(k_2, < t_2, i_2, f_2 >), \dots, (k_j, < t_j, i_j, f_j >), \dots\}, \qquad (1)$$

which means that in the set $M$ the element $x$ is repeated $k_1$ times with the neutrosophic components $<t_1,i_1,f_1>$, and $k_2$ times with the neutrosophic components $<t_2,i_2,f_2>$), ..., $k_j$ times

with the neutrosophic components $<t_j,i_j,f_j>)$, ... and so on. Of course, $<t_p,i_p,f_p>) \neq <t_r,i_r,f_r>$ for $p \neq r$.

For example, with respect to the above neutrosophic multiset $Q$,

$nm(a) = \{(2, <0.6,0.3,0.2>), (1, <0.5,0.4,0.1>)\}$.

Neutrosophic multiset is used in time series, and in representing instances of the physical process at different times, since its neutrosophic components change in time.

## *X.3. Neutrosophic Complex Multiset*

Let $\mathcal{U}$ be a universe of discourse, and $\mathcal{S} \subset \mathcal{U}$.

A *Neutrosophic Complex Multiset* $\mathcal{S}$ is a neutrosophic complex set, which has one or more elements that repeat either with the same complex neutrosophic components, or with different other complex neutrosophic components.

*Example of Neutrosophic Complex Set.*

$$B_1 = \begin{Bmatrix} a\left(0.3e^{j(0.2)}, 0.1e^{j(0.1)}, 0.8e^{j(0.5)}\right), \\ b\left(0.5e^{j(0.4)}, 0.2e^{j(0.3)}, 0.1e^{j(0.2)}\right) \end{Bmatrix}$$

is a neutrosophic complex set.

*Examples of Neutrosophic Complex Multiset.*

$$B_2 = \begin{Bmatrix} a\left(0.3e^{j(0.2)}, 0.1e^{j(0.1)}, 0.8e^{j(0.5)}\right), \\ a\left(0.3e^{j(0.2)}, 0.1e^{j(0.1)}, 0.8e^{j(0.5)}\right) \end{Bmatrix}$$

is a neutrosophic complex multiset because the element $a$ repeats with the same neutrosophic complex components.

$$B_3 = \begin{Bmatrix} a\left(0.3e^{j(0.2)}, 0.1e^{j(0.1)}, 0.8e^{j(0.5)}\right), \\ a\left(0.4e^{j(0.3)}, 0.2e^{j(0.1)}, 0.7e^{j(0.4)}\right), \\ b\left(0.5e^{j(0.4)}, 0.2e^{j(0.3)}, 0.1e^{j(0.2)}\right) \end{Bmatrix}.$$

is a neutrosophic complex multiset because the element $a$ repeats, but with different neutrosophic complex components.

$$B_4 = \begin{Bmatrix} a\left(0.3e^{j(0.2)}, 0.1e^{j(0.1)}, 0.8e^{j(0.5)}\right), \\ a\left(0.3e^{j(0.2)}, 0.1e^{j(0.1)}, 0.8e^{j(0.5)}\right), \\ a\left(0.7e^{j(0.6)}, 0.2e^{j(0.1)}, 0.1e^{j(0.0)}\right), \\ b\left(0.7e^{j(0.2)}, 0.0e^{j(0.3)}, 0.4e^{j(0.2)}\right) \end{Bmatrix}.$$

is a neutrosophic complex multiset because the element "$a$" repeats once with the same neutrosophic components, and afterwards with different neutrosophic components.

Similarly, we define the *Neutrosophic Complex Multiplicity Function*:

$ncm: \mathcal{U} \to N = \{1, 2, 3, \dots\}$

for $a \in \mathcal{S}$ one has

$$ncm(a): \{(k_1, < t_1 e_{j\alpha_1}, i_1 e_{j\beta_1}, f_1 e_{j\gamma_1} >), (k_2, < t_2 e_{j\alpha_2}, i_2 e_{j\beta_2}, f_2 e_{j\gamma_2} >), \dots, (k_n, < t_n e_{j\alpha_n}, i_n e_{j\beta_n}, f_n e_{j\gamma_n} >), \dots\}.$$

Whence, a neutrosophic complex multiset $\mathcal{S}$ can be written as $(\mathcal{S}, ncm(a))$ or $\{(a, ncm(a)), \text{for } a \in \mathcal{S}\}$.

**CHAPTER XI**

# *XI.1. Classical Logic and Neutrosophic Logic. Answers to K. Georgiev.*

## *Abstract.*

In this paper, we make distinctions between Classical Logic (where the propositions are 100% true, or 100% false) and the Neutrosophic Logic (where one deals with partially true, partially indeterminate and partially false propositions) in order to respond to K. Georgiev [1]'s criticism. We recall that if an axiom is true in a classical logic system, it is not necessarily that the axiom be valid in a modern (fuzzy, intuitionistic fuzzy, neutrosophic etc.) logic system.

## *XI.1.1. Single Valued Neutrosophic Set.*

We read with interest the paper [1] by K. Georgiev. The author asserts that he proposes "a general simplification of the Neutrosophic Sets a subclass of theirs, comprising of elements of $R^3$", but this was actually done before, since the first world publication on neutrosophics [2].

The simplification that Georgiev considers, is called single-valued neutrosophic set.

The single valued neutrosophic set was introduced for the first time by us [Smarandache, [2], 1998].

Let $n = t + i + f$. (1)

In Section 3.7, "Generalizations and Comments", [pp. 129, last edition online], from this book [2], we wrote:

*"Hence, the neutrosophic set generalizes:*

*- the intuitionistic set, which supports incomplete set theories (for $0 < n < 1$; $0 \leq t, i, f \leq 1$) and incomplete known elements belonging to a set;*

*- the fuzzy set (for $n = 1$ and $i = 0$, and $0 \leq t, i, f \leq 1$);*

*- the classical set (for $n = 1$ and $i = 0$, with $t, f$ either 0 or 1);*

*- the paraconsistent set (for $n > 1$, with all $t, i, f < 1$);*

*- the faillibilist set ($i > 0$);*

*- the dialetheist set, a set M whose at least one of its elements also belongs to its complement*

*C(M); thus, the intersection of some disjoint sets is not empty;*

*- the paradoxist set (t = f = 1);*

*- the pseudoparadoxist set (0 < i < 1; t =1 and f > 0 or t > 0 and f = 1);*

*- the tautological set (i, f < 0).”*

It is clear that we have worked with single-valued neutrosophic sets, we mean that *t, i, f* were explicitly real numbers from *[0, 1].*

See also (Smarandache, [3], 2002, p. 426).

More generally, we have considered that: *t* varies in the set *T, i* varies in the set *I*, and *f* varies in the set *F*, but in the same way taking crisp numbers *n = t + i + f,* where all *t, i, f* are single (crisp) real numbers in the interval *[0, 1].* See [2] pp. 123-124, and [4] pp. 418-419.

Similarly in *The Free Online Dictionary of Computing [FOLDOC],* 1998, updated in 1999, edited by Denis Howe [3].

Unfortunately, Dr. Georgiev in 2005 took into consideration only the neutrosophic publication [6] from year 2003, and he was not aware of

previous publications [2, 3, 4] on the neutrosophics from the years 1998 - 2002.

The misunderstanding was propagated to other authors on neutrosophic set and logic, which have considered that Haibin Wang, Florentin Smarandache, Yanqing Zhang, Rajshekhar Sunderraman (2010, [5]) have defined the single valued neutrosophic set.

## *XI.1.2. Standard and Non-Standard Real Subsets.*

Section 3 of paper [1] by Georgiev is called "Reducing Neutrosophic Sets to Subsets of $R^3$". But this was done already since 1998. In our Section 0.2, [2], p. 12, we wrote:

*"Let T, I, F be standard or non-standard real subsets...".*

"Standard real subsets", which we talked about above, mean just the classical real subsets.

We have taken into consideration the non-standard analysis in our attempt to be able to describe the *absolute truth* as well [i.e. truth in all possible worlds, according to Leibniz's denomination, whose neutrosophic truth value is equal

to $1^+ = 1+\varepsilon$, where $\varepsilon$ is a tiny number $> 0$ ], and *relative truth* [i.e. truth in at least one world, whose truth value is equal to *1*]. Similarly for absolute / relative indeterminacy and absolute / relative falsehood.

We tried to get a definition as general as possible for the neutrosophic logic (and neutrosophic set respectively), including the propositions from a philosophical point of [absolute or relative] view.

Of course, in technical and scientific applications we do not consider non-standard things, we take the classical unit interval *[0, 1]* only, while *T, I, F* are classical real subsets of it.

In Section 0.2, Definition of Neutrosophic Components [2], 1998, p. 12, we wrote:

> *"The sets T, I, F are not necessarily intervals, but may be any real sub-unitary subsets: discrete or continuous; single-element, finite, or (countable or uncountable) infinite; union or intersection of various subsets; etc.*
>
> *They may also overlap. The real subsets could represent the relative errors in determining t, i,*

*f (in the case when the subsets T, I, F are reduced to points)."*

So, we have mentioned many possible real values for *T, I, F*. Such as: each of *T, I, F* can be "single-element" {as Georgiev proposes in paper [1]}, "interval" {developed later in [7], 2005, and called interval-neutrosophic set and interval-neutrosophic logic respectively}, "discrete" [called hesitant neutrosophic set and hesitant neutrosophic logic respectively] etc.

## *XI.1.3. Degrees of Membership > 1 or < 0 of the Elements.*

In Section 4 of paper [1], Georgiev says that: "Smarandache has adopted Leibniz's 'worlds' in his work, but it seems to be more like a game of words."

As we have explained above, "Leibniz's worlds" are not simply a game of words, but they help making a distinction *in philosophy* between absolute and relative truth / indeterminacy / falsehood respectively. {In technics and science yes they are not needed.}

Besides absolute and relative, the non-standard values or hyper monads $(^{-}0$ and $1^{+})$ have permitted us to introduce, study and show applications of the neutrosophic overset (when there are elements into a set whose real (standard) degree of membership is $> 1$), neutrosophic underset (when there are elements into a set whose real degree of membership is $< 0$), and neutrosophic offset (when there are both elements whose real degree of membership is $> 1$ and other elements whose real degree of membership is $< 0$). Check the references [8-11].

## *XI.1.4. Neutrosophic Logic Negations.*

In Section 4 of the same paper [1], Georgiev asserts that "according to the neutrosophic operations we have

$$\neg\neg A = A \tag{2}$$

and since

$$\neg\neg A \neq A \tag{3}$$

is just the assumption that has brought intuitionism to life, the neutrosophic logic could not be a generalization of any Intuitionistic logic."

First of all, Georgiev's above assertation is partially true, partially false, and partially indeterminate (as in the neutrosophic logic).

In neutrosophic logic, there is a class of neutrosophic negation operators, not only one. For some neutrosophic negations the equality (2) holds, for others it is invalid, or indeterminate.

Let *A(t, i, f)* be a neutrosophic proposition A whose neutrosophic truth value is *(t, i, f)*, where *t, i, f* are single real numbers of *[0, 1]*. We consider the easiest case.

a) For examples, if the neutrosophic truth value of $\neg A$, the negation of *A*, is defined as:

*(1-t, 1-i, 1-f) or (f, i, t) or (f, 1-i, t)* (4)

then the equality (2) is valid.

b) Other examples, if the neutrosophic truth value of $\neg A$, the negation of *A*, is defined as:

*(f, (t+i+f)/3, t)* or *(1-t, (t+i+f)/3, 1-f)* (5)

then the equality (2) is invalid, as in intuitionistic fuzzy logic, and as a consequence the inequality (3) holds.

c) For the future new to be designed / invented neutrosophic negations (needed / adjusted for new applications) we do not know {so (2) has also a percentage of indeterminacy}.

## *XI.1.5. Degree of Dependence and Independence between (Sub)Components.*

In Section 4 of [1], Georgiev also asserts that "The neutrosophic logic is not capable of maintaining modal operators, since there is no normalization rule for the components *T, I, F*". This is also partially true, and partially false.

In our paper [12] about the dependence / independence between components, we wrote that:

*"For single valued neutrosophic logic, the sum of the components t+i+f is:*

*$0 \leq t+i+f \leq 3$ when all three components are 100% independent;*

*$0 \leq t+i+f \leq 2$ when two components are 100% dependent, while the third one is 100% independent from them;*

*$0 \leq t+i+f \leq 1$ when all three components are 100% dependent.*

*When three or two of the components t, i, f are 100% independent, one leaves room for incomplete information (therefore the sum $t+i+f < 1$), paraconsistent and contradictory information ($t+i+f > 1$), or complete information ($t+i+f = 1$).*

*If all three components t, i, f are 100% dependent, then similarly one leaves room for incomplete information ($t+i+f < 1$), or complete information ($t+i+f = 1$).”*

Therefore, for complete information the normalization to *1, 2, 3* or so respectively {see our paper [12] for the case when one has degrees of dependence between components or between subcomponents (for refined neutrosophic set respectively) which are different from 100% or 0%} is done.

But, for incomplete information and paraconsistent information, in general, the normalization is not done.

Neutrosophic logic is capable of maintaining modal operators. The connection between Neutrosophic Logic and Modal Logic will be shown in a separate paper, since it is much longer, called Neutrosophic Modal Logic (under press).

## *XI.1.6. Definition of Neutrosophic Logic.*

In Section 5, paper [1], it is said: "Apparently there isn't a clear definition of truth value of the neutrosophic formulas." The author is right that "apparently", but in reality, the definition of neutrosophic logic is very simple and common sense:

In neutrosophic logic a proposition *P* has a degree of truth *(T);* a degree of indeterminacy *(I)* that means neither true nor false, or both true and false, or unknown, indeterminate; and a degree of falsehood *(F);* where *T, I, F* are subsets (either real numbers, or intervals, or any subsets) of the interval *[0, 1].*

What is unclear herein?

In a soccer game, as an easy example, between two teams, Bulgaria and Romania, there is a degree of truth about Bulgaria winning, degree of

indeterminacy (or neutrality) of tie game, and degree of falsehood about Bulgaria being defeated.

## *XI.1.7. Neutrosophic Logical Systems.*

a) Next sentence of Georgiev is "in every meaningful logical system if A and B are sets (formulas) such that A ⊆ B then B ' A, i.e. when B is true then A is true." (6)

In other words, when B → A (B implies A), and B is true, then A is true.

This is true for the Boolean logic where one deals with *100%* truths, but in modern logics we work with partial truths.

If an axiom is true in the classical logic, it does not mean that that axiom has to be true in the modern logical system. Such counter-example has been provided by Georgiev himself, who pointed out that the law of double negation {equation (2)}, which is valid in the classical logic, is not valid any longer in intuitionistic fuzzy logic.

A similar response we have with respect to his above statement on the logical system axiom (6): it is partially true, partially false, and partially

indeterminate. All depend on the types of chosen neutrosophic implication operators.

In neutrosophic logic, let's consider the neutrosophic propositions $A(t_A, i_A, f_A)$ and $B(t_B, i_B, f_B)$, and the neutrosophic implication:

$$B(t_B, i_B, f_B) \rightarrow A(t_A, i_A, f_A), \quad (7)$$

that has the neutrosophic truth value

$$(B \rightarrow A)(t_{B\rightarrow A}, i_{B\rightarrow A}, f_{B\rightarrow A}). \quad (8)$$

Again, we have a class of many neutrosophic implication operators, not only one; see our publication [13], 2015, pp. 79-81.

Let's consider one such neutrosophic implication for single valued neutrosophic logic:

$(B \rightarrow A)(t_{B\rightarrow A}, i_{B\rightarrow A}, f_{B\rightarrow A})$ is equivalent to $B(t_B, i_B, f_B) \rightarrow A(t_A, i_A, f_A)$

which is equivalent to $\neg B(f_B, 1-i_B, t_B) \vee A(t_A, i_A, f_A)$

which is equivalent to $(\neg B \vee A)(\max\{f_B, t_A\}, \min\{1-i_B, i_A\}, \min\{t_B, f_A\})$. (9)

Or: $(t_{B\rightarrow A}, i_{B\rightarrow A}, f_{B\rightarrow A}) = (\max\{f_B, t_A\}, \min\{1-i_B, i_A\}, \min\{t_B, f_A\})$. (10)

Now, a question arises: what does "(B →) A is true" mean in fuzzy logic, intuitionistic fuzzy logic, and respectively in neutrosophic logic?

Similarly for the "B is true", what does it mean in these modern logics? Since in these logics we have infinitely many truth values t(B) ∈ (0, 1); {we made abstraction of the truth values 0 and 1, which represent the classical logic}.

b) Theorem 1, by Georgiev, "Either A ⊞ k(A) [i.e. A is true if and only if k(A) is true] or the neutrosophic logic is contradictory."

We prove that his theorem is a nonsense.

First at all, the author forgets that when he talks about neutrosophic logic he is referring to a modern logic, not to the classical (Boolean) logic. The logical propositions in neutrosophic logic are partially true, in the form of *(t, i, f),* not totally 100% true or *(1, 0, 0).* Similarly for the implications and equivalences, they are not classical (i.e. *100%* true), but partially true {i.e. their neutrosophic truth values are in the form of *(t, i, f)* too}.

- The author starts using the previous classical logical system axiom (6), i.e.

"since $k(A) \subseteq A$ we have $A \Rightarrow k(A)$" meaning that $A \rightarrow$ k(A) and when A is true, then k(A) is true.

- Next Georgiev's sentence: "Let assume k(A) be true and assume that A is not true".

The same comments as above:

What does it mean in fuzzy logic, intuitionistic fuzzy logic, and neutrosophic logic that a proposition is true? Since in these modern logics we have infinitely many values for the truth value of a given proposition. Does, for example, t(*k(A)*) = 0.8 {i.e. the truth value of *k(A)* is equal to 0.8}, mean that *k(A)* is true?

If one takes t(*k(A)*) = 1, then one falls in the classical logic.

Similarly, what does it mean that proposition A is not true? Does it mean that its truth value

t(A) = 0.1 or in general t(A) < 1 ? Since, if one takes t(A) = 0, then again we fall into the classical logic.

The author confuses the classical logic with modern logics.

- In his "proof" he states that "since the Neutrosophic logic is not an intuitionistic one, $\neg$ A should be true

leading to the conclusion that $k(\neg A) = \neg k(A)$ is true".

For the author an "intuitionistic logic" means a logic that invalidates the double negation law {equation (3)}. But we have proved before in Section 4, of this paper, that depending on the type of neutrosophic negation operator used, one has cases when neutrosophic logic invalidates the double negation law [hence it is "intuitionistic" in his words], cases when the neutrosophic logic does not invalidate the double negation law {formula (2)}, and indeterminate cases {depending on the new possible neutrosophic negation operators to be design in the future}.

- The author continues with "We found that $k(A) \wedge \neg k(A)$ is true which means that the simplified neutrosophic logic is contradictory."

Georgiev messes up the classical logic with modern logic. In classical logic, indeed $k(A) \wedge \neg k(A)$ is false, being a contradiction.

But we are surprised that Georgiev does not know that in modern logic we may have $k(A) \wedge \neg$

$k(A)$ that is not contradictory, but partially true and partially false.

For example, in fuzzy logic, let's say that the truth value (t) of $k(A)$ is $t(k(A)) = 0.4$, then the truth value of its negation, $\neg k(A)$, is $t(\neg k(A)) = 1 - 0.4 = 0.6$.

Now, we apply the t-norm "min" in order to do the fuzzy conjunction, and we obtain:

$t(k(A) \wedge \neg k(A)) = \min\{0.4, 0.6\} = 0.4 \neq 0$.

Hence, $k(A) \wedge \neg k(A)$ is not a contradiction, since its truth value is 0.4, not 0.

Similarly in intuitionistic fuzzy logic.

The same in neutrosophic logic, for example:

Let the neutrosophic truth value of $k(A)$ be (0.5, 0.4, 0.2), that we denote as:

$k(A)(0.5, 0.4, 0.2)$, then its negation $\neg k(A)$ will have the neutrosophic truth value:

$\neg k(A)(0.2, 1\text{-}0.4, 0.5) = \neg k(A)(0.2, 0.6, 0.5)$.

Let's do now the neutrosophic conjunction:

$k(A)(0.5, 0.4, 0.2) \wedge \neg k(A)(0.2, 0.6, 0.5) = (k(A) \wedge \neg k(A))(\min\{0.5, 0.2\}, \max\{0.4, 0.6\}, \max\{0.2, 0.5\}) = (k(A) \wedge \neg k(A))(0.2, 0.6, 0.5)$.

In the same way, $k(A) \wedge \neg k(A)$ is not a contradiction in neutrosophic logic, since its neutrosophic truth value is (0.2, 0.6, 0.5), which is different from (0, 0, 1) or from (0, 1, 1).

Therefore, Georgiev's "proof" that the simplified neutrosophic logic [ = single valued neutrosophic logic] is a contradiction has been disproved!

His following sentence, "But since the simplified neutrosophic logic is only a subclass of the neutrosophic logic, then the neutrosophic logic is a contradiction" is false. Simplified neutrosophic logic is indeed a subclass of the neutrosophic logic, but he did not prove that the so-called simplified neutrosophic logic is contradictory (we have showed above that his "proof" was wrong).

## *XI.1.8. Conclusion.*

We have showed in this paper that Georgiev's critics on the neutrosophic logic are not founded. We made distinctions between the Boolean logic systems and the neutrosophic logic systems.

Neutrosophic logic is developing as a separate entity with its specific neutrosophic logical systems, neutrosophic proof theory and their applications.

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# *XI.2. Neutrosophic Tautological, Contradictional, and Provability Tresholds, and Neutrosophic Completeness. Answers to U. Rivieccio*

First, I'd like to thank Dr. Rivieccio for his detailed study on neutrosophic logic and related topics [1].

## *XI.2.1. Belnap's logic system*

*Belnap's logic system of four values*: (1,0) corresponding to *truth*, (0,1) corresponding to *falsehood*, (0,0) corresponding to *unknown*, (1,1) corresponding to *contradiction* are actually a particular case of the Refined Neutrosophic Logic (2013), where the *neutrosophic components* $(t, i, f)$, with $t$ = true, $i$ = indeterminacy, and $f$ = false, can be refined [6] as follows:

$$(t_1, t_2, \dots, t_p; i_1, i_2, \dots, i_r; f_1, f_2, \dots, f_s)$$

where p, r, s are integers ≥1, and

$t_j$, $1 \le j \le p$, is a sub-truth,

$i_k$, $1 \le k \le r$, is a sub-indeterminacy, and

$f_l$, $1 \le l \le s$, is a sub-falsehood.

Taking the simple case, when the neutrosophic components $(t, i, f)$ and the neutrosophic subcomponents

$$\left(\{t_j\}_{j\in\{1,2,\ldots,p\}}, \{i\}_{k\in\{1,2,\ldots,r\}}, \{f_l\}_{l\in\{1,2,\ldots,s\}}\right)$$

are single-valued numbers in [0,1] , one has that:

$$0 \leq t + i + f \leq 3 \quad (1)$$

while: (2)

$$0 \leq \sum_{j=1}^{p} t_j + \sum_{k=1}^{r} i_k + \sum_{l=1}^{s} f_l \leq p + r + s.$$

Therefore, Belnap's logic system is a refined neutrosophic logic of the form: $(t, i_1, i_2, f)$, where $t$ = truth, $f$ = falsehood, $i_1$= unknown (or first sub-indeterminacy), and $i_2$= contradiction (or second sub-indeterminacy).

## *XI.2.2. Kleene's three-valued logic,*

*Kleene's three-valued logic* with (0,0) as *undefined*, (0,1) as *falsehood*, and (1,0) as *truth*, is also a neutrosophic logic (t, i, f), where the indeterminacy *i* is perceived as undefined.

## *XI.2.3. Paraconsistent Logics*

As Rivieccio has observed, the neutrosophic logic can catch the paraconsistent logics, since

$t + i + f$ may be $> 1$,

so, we may have conflicting information.

Neither fuzzy logic, nor Atanassov's intuitionistic fuzzy logic allowed the sum of components to exceed 1.

In fuzzy logic: $t + f = 1$, and in intuitionistic fuzzy logic: $t + f \leq 1$, leaving room for some hesitancy: $1 - (t + f)$, which is denoted as indeterminacy in neutrosophic logic.

### *XI.2.4. Incomplete Logics*

Even more, the neutrosophic logic allowed the sum of the components to be strictly less than 1,

$t + i + f < 1$

for incomplete logics, i.e. logics where there is missing or incomplete information.

Again, neither fuzzy logic nor intuitionistic fuzzy logic allowed the sum of the components to be strictly less than 1.

### *XI.2.5. Set-Valued Neutrosophic Logic*

While both fuzzy logic and intuitionistic fuzzy logic have extended their fields from "crisp values" to "interval values" assigned to their

components, defining the interval-valued fuzzy logic and respectively interval-valued intuitionistic fuzzy logic, neutrosophic logic went further and considered set-valued neutrosophic logic, where the components *t, i, f* are not necessarily intervals, but in general subsets of the unit interval [0,1].

## *XI.2.6. Non-standard Set-Valued Neutrosophic Logic*

Even more, and not done in fuzzy logic nor in intuitionistic fuzzy logic, from a philosophical point of view the neutrosophic logic can distinguish between an *absolute truth*, which is a truth in all possible worlds (according to Leibniz), whose neutrosophic value is denoted by

$1^+ = 1 + \varepsilon,$

where $\varepsilon > 0$ is a tinny number, and *relative truth*, which is truth in at least one world, whose neutrosophic value is denoted by 1.

Similarly, for absolute/ relative indeterminacy and respectively falsehood.

*XI.2.7.* Regarding the interpretations of ⟨1,0,0⟩ as truth, ⟨0,1,1⟩ as contradiction, and ⟨1,1,1⟩ as paradox in the book [2], Rivieccio writes that "this does not seem quite convincing, since intuitively it is not clear why for instance ⟨0,1,1⟩ should be contradiction more than ⟨0,0,1⟩."

If we consider *t, i, f* as singletons, then the interpretations are the following, as in refined neutrosophic logic

$\langle t, i_1, i_2, i_3, i_4, i_5, f \rangle$:

$t = \langle 1,0,0 \rangle$ truth;

$$\left.\begin{matrix} i_1 = \langle 0,1,0 \rangle \\ i_2 = \langle 1,1,0 \rangle \\ i_3 = \langle 1,0,1 \rangle \\ i_4 = \langle 0,1,1 \rangle \\ i_5 = \langle 1,1,1 \rangle \end{matrix}\right\} \text{ sub-indeterminacies}$$

$f = \langle 0,0,1 \rangle$ falsehood;

where

$i_1$ = pure indeterminacy;

$i_2$ = truth-indeterminacy confusion;

$i_3$ = contradiction (true & false simultaneously);

$i_4$ = indeterminacy-falsehood confusion;

$i_5$ = paradox (true & false & indeterminate simultaneously).

Rivieccio continues: "We would suggest a more cautious interpretation, i.e. to consider the indeterminacy degree as a measure of the reliability (conversely, the imprecision, error, etc.) of a certain source of information." We fully agree with his suggestion, that's how is in our previous refined neutrosophic logic the indeterminacy, and we split it into types of sub-indeterminacies, explicitly described.

In general, we can split the indeterminacy degree into: degree of vagueness, degree of imprecision, degree of error, degree of conflicting, degree of incompleteness, and so on – depending on the needed application (or problem) to solve.

## *XI.2.8. Neutrosophic Negation Operator*

We agree with Rivieccio that the first neutrosophic negation operator [4] that we proposed starting from 1995, defined as:

$$\overset{\neg}{N}(t,i,f) = (1-t, 1-i, 1-f) \qquad (3)$$

is not the best, although a straightforward extension of the most common fuzzy logic negation operator.

In the meantime, more neutrosophic negation operators have been proposed by various authors and us, forming a class of neutrosophic negation operators.

We agree that

$$\neg(t, i, f) = (f, i, t) \tag{4}$$

is the best neutrosophic negation operator (Ashbacher), [3] so far.

## *XI.2.9. Neutrosophic Conjunction Operator*

Similarly, our first neutrosophic conjunction [4]

$$(c_1)\ \langle t_1, i_1, f_1\rangle \overset{\wedge}{_N} \langle t_2, i_2, f_2\rangle = \langle t_1 t_2, i_1 i_2, f_1 f_2\rangle \tag{5}$$

is less accurate (we agree with Rivieccio), than:

$$(c_2)\ \langle t_1, i_1, f_1\rangle \overset{\wedge}{_N} \langle t_2, i_2, f_2\rangle =$$

$$\langle min\{t_1, t_2\}, min\{i_1, i_2\}, max\{f_1, f_2\}\rangle \tag{6}$$

or $(c_3)\ \langle t_1, i_1, f_1\rangle \overset{\wedge}{_N} \langle t_2, i_2, f_2\rangle =$

$$\langle min\{t_1, t_2\}, max\{i_1, i_2\}, max\{f_1, f_2\}\rangle \tag{7}$$

as defined by Ashbacher [3].

As the truth ($t$) is considered a positive quality, while the indeterminacy ($i$) and the falsehood ($f$) are negative qualities, whatever operation we do to $t_1$ and $t_2$ we have to do the opposite to $i_1, i_2$ and respectively to $f_1, f_2$.

Therefore $(c_3)$ is the best.

However, today (June 2017) the most general classes of neutrosophic conjunction operators have the forms:

$$\langle t_1, i_1, f_1\rangle \overset{\wedge}{}_N \langle t_2, i_2, f_2\rangle = \langle t_1 \overset{\wedge}{}_F t_2, i_1 \overset{\vee}{}_F i_2, f_1 \overset{\vee}{}_F f_2\rangle \qquad (8)$$

or

$$\langle t_1, i_1, f_1\rangle \overset{\wedge}{}_N \langle t_2, i_2, f_2\rangle = \langle t_1 \overset{\wedge}{}_F t_2, i_1 \overset{\wedge}{}_F i_2, f_1 \overset{\vee}{}_F f_2\rangle \qquad (9)$$

where $\overset{\wedge}{}_F$ is a fuzzy t-norm, for examples:

$$a \overset{\wedge}{}_F b = a \cdot b; \qquad (10)$$

or

$$a \overset{\wedge}{}_F b = min\{a, b\}; \qquad (11)$$

or

$$a \overset{\wedge}{}_F b = max\{a + b - 1, 0\}, \qquad (12)$$

or others;

while $\overset{\vee}{}_F$ is a fuzzy t-conorm, for examples:

$$a \overset{\vee}{_F} b = a + b - ab; \tag{13}$$

or

$$a \overset{\vee}{_F} b = max\{a, b\}; \tag{14}$$

or

$$a \overset{\vee}{_F} b = min\{a + b, 1\}, \tag{15}$$

or others.

## *XI.2.10. Neutrosophic Disjunction Operator*

In the same way as we responded for the neutrosophic negation and conjunction operators, our first neutrosophic disjunction operator (1995) in [4]:

$$(D_1)\ \langle t_1, i_1, f_1\rangle \overset{\vee}{_N} \langle t_2, i_2, f_2\rangle =$$

$$\langle t_1 + t_2 - t_1 t_2, i_1 + i_2 - i_1 i_2, f_1 + f_2 - f_1 f_2\rangle \tag{16}$$

is less accurate, since indeed "should not treat the truth, indeterminacy and falsity components in the same way" [Rivieccio] is right.

In a pessimistic way, we had proposed later on to treat the indeterminacy and falsity components in the same way (as qualitatively negative components), while the truth component in an

opposite was (as qualitatively positive components).

But in an optimistic way, the truth and indeterminacy can be considered in the same way, while the falsity is an opposite way.

The definitions by Ashbacher (2002) are more accurate:

$$(D_2)\ \langle t_1, i_1, f_1\rangle \overset{\vee}{}_N \langle t_2, i_2, f_2\rangle =$$

$$\langle max\{t_1, t_2\}, max\{i_1, i_2\}, min\{f_1, f_2\}\rangle \qquad (17)$$

$$(D_3)\ \langle t_1, i_1, f_1\rangle \overset{\vee}{}_N \langle t_2, i_2, f_2\rangle =$$

$$\langle max\{t_1, t_2\}, min\{i_1, i_2\}, min\{f_1, f_2\}\rangle \qquad (18)$$

However, today (June 2017) the most general classes of neutrosophic disjunction operators have the forms (dualistic to the neutrosophic conjunction operators):

$$\langle t_1, i_1, f_1\rangle \overset{\vee}{}_N \langle t_2, i_2, f_2\rangle = \langle t_1 \overset{\vee}{}_F t_2, i_1 \overset{\wedge}{}_F i_2, f_1 \overset{\wedge}{}_F f_2\rangle \qquad (19)$$

or

$$\langle t_1, i_1, f_1\rangle \overset{\vee}{}_N \langle t_2, i_2, f_2\rangle = \langle t_1 \overset{\vee}{}_F t_2, i_1 \overset{\vee}{}_F i_2, f_1 \overset{\wedge}{}_F f_2\rangle. \qquad (20)$$

## *XI.2.11. Incomplete Neutrosophic Logic*

The neutrosophic logic where the sum of the components $t + i + f \leq 1$ should be called *incom-*

*plete neutrosophic logic*, not *intuitionistic neutrosophic logic* (Ashbacher, [3]), in order to avoid the confusion of meaning of the word "intuitionistic".

## *XI.2.12. Neutrosophic Implication*

We have defined a class of neutrosophic implication connectives

$$\langle t_1, i_1, f_1\rangle \underset{N}{\rightarrow} \langle t_2, i_2, f_2\rangle, \tag{21}$$

in {Smarandache [7], pp. 79-81} as an adaptation from the classical logic implication, and from fuzzy logic (S-implication):

$$(I_1)\ \ \neg_N\langle t_1, i_1, f_1\rangle \overset{\vee}{}_N \langle t_2, i_2, f_2\rangle. \tag{22}$$

Rivieccio presents the below neutrosophic implication:

$$(I_2)\ \left\langle \begin{matrix} min\{1,1-t_1+t_2\}, max\{0, i_2 - i_1\}, \\ max\{0, f_2 - f_1\} \end{matrix} \right\rangle \tag{23}$$

as an extension of Lukasiewicz logic's implication connective.

He criticizes Ashbacher's neutrosophic system for having no "tautologies", meaning that "there is no sentence **p** such that $v(p) = \langle 1, 0, 0\rangle$ for every neutrosophic valuation $v$".

First of all, a *neutrosophic valuation* is an approximation of the neutrosophic truth value of a proposition $p$. A source $A$ approximates in one way, $v_A(p) = \langle t_A, i_A, f_A \rangle$ while another source $B$ approximates in a different way, $v_{B(p)} = \langle t_B, i_B, f_B \rangle$ .

In fuzzy, intuitionistic fuzzy, and neutrosophic logics, we deal with estimations, approximations, and subjectivity. The aggregations / connectives / rules of inference do approximate calculations. The indeterminacy makes a difference in the multiple-valued logic laws.

## *XI.2.13. Neutrosophic "Tautology"*

We have tautologies (propositions whose truth-value is 1) in classical (Boolean) logic.

But what is "tautology" in fuzzy, intuitionistic fuzzy, and neutrosophic logics, where we work with partial truth ($0 < t < 1$) ?

Can we say that a proposition $p$, whose truth-value is 0.8, is a tautology or not?

An idea would be to consider a *neutrosophic tautological threshold* $\tau(t_\tau, i_\tau, f_\tau)$, and each proposition $p$ that is equal or above this neutrosophic tautological threshold should be

considered a *neutrosophic tautology*, while if it is below it should be not.

How to establish such threshold?

Of course, this should be handled by experts upon the application or problem they need to solve.

The two neutrosophic relationships $\leq_{N_1}$ and $\leq_{N_2}$ presented in Rivieccio's paper are partial order relationships: (24-25)

$(N_1)$ $(t_1, i_1, f_1) \leq_{N_1} (t_2, i_2, f_2)$ iff $t_1 \leq t_2, i_1 \leq i_2, f_1 \geq f_2$;

$(N_2)$ $(t_1, i_1, f_1) \leq_{N_2} (t_2, i_2, f_2)$ iff $t_1 \leq t_2, i_1 \geq i_2, f_1 \geq f_2$.

We prefer to use $(N_2)$, since the sense inequalities, for $(i)$ and for $(f)$ should be the same $(\geq)$, while that for $(t)$ should be the opposite $(\leq)$.

We'll further denote it simply by $\leq_N$.

## *XI.2.14. Neutrosophic Propositional Logic*

Let $\lambda$ be the set of all neutrosophic propositions $P$, where the neutrosophic validation (truth-value) of $P$ is $(t_P, i_P, f_P)$, with $t_P, i_P, f_P \in [0,1]$.

We consider the simplest case, when the neutrosophic components are single-valued numbers.

The cases when $t_P, i_P, f_P$ are intervals or in general subsets of [0,1] are straight-forwarded generalizations of single-valued neutrosophic components.

Let the *neutrosophic tautological threshold* be $\tau(t_\tau, i_\tau, f_\tau)$, determined by the neutrosophic experts with respect to an application to solve, where $t_\tau, i_\tau, f_\tau \in [0,1]$.

The neutrosophic validation function:

$$v_N: \lambda \to [0,1]^3, \text{ with } v_N(P) = (t_P, i_P, f_P) \quad (26)$$

and (27)

$$\lambda = \left\{ \begin{matrix} P, \text{where } P \textit{ is a } \text{neutrosophic proposition,} \\ v_N(P) = (t_P, i_P, f_P) \in [0,1]^3 \end{matrix} \right\}.$$

The set $\lambda$ is split into three subsets:

a) The set of *neutrosophic tautologies* (or neutrosophically true propositions with respect to neutrosophic tautological threshold $\tau$):

$$Taut = \{P \in \lambda, v_N(P) \geq v_N(\tau)\}. \quad (28)$$

b) The set of *neutrosophic non-tautologies* (or neutrosophically false propositions with respect to the neutrosophic tautological threshold $\tau$):

$$NonTaut = \{P \in \lambda, v_N(P) <_N v_N(\tau)\} \quad (29)$$

c) The set of *neutrosophic undecided propositions* (or neutrosophically neither true nor false propositions with respect to the neutrosophic tautological threshold $\tau$): (30)

$$Undecided = \{P \in \lambda, v_N(P) N^{\not\geq} v(\tau) \text{ and } v_N \not\leq_N v(\tau)\}.$$

Since the neutrosophic inequality $\leq_N$ establishes only a partial order on $\lambda$, therefore $\lambda$ is a *neutrosophic poset* (partial ordered set), one has in $\lambda$ neutrosophic propositions, let's say $P_1$ and $P_2$, such that neiter $v_N(P_1) \leq_N v(P_2)$, nor $v_n(P_1) N^{>} v_N(P)$.

## *XI.2.15. Neutrosophic "Completeness"*

Many definitions of completeness exist, with respect to various fields of knowledge.

a) In classical logic, if a proposition $P$ cannot be derived from the system's axioms, it gives rise to a contradiction.

But what is a "contradiction" in fuzzy, intuitionistic fuzzy, or neutrosophic logics?

If $P$ is such that its fuzzy validation $(v_F)$ is

$v_F(P) = 0.5$,

then $v_F({}_F\neg P) = 1 - 0.5 = 0.5$,

and $v_F\left(P \underset{F}{\wedge} (_F\neg P)\right) = 0.5$, so $P \underset{F}{\wedge} (_F\neg P)$ is not a fuzzy contradiction.

Similarly, if the intuitionistic fuzzy logic validation ($v_{IF}$) of P is, for example,

$v_{IF}(P) = (0.5, 0.5)$,

then $v_{IF}(_{IF}\neg P) = (0.5, 0.5)$,

and $v_{IF}\left(P \underset{IF}{\wedge} (_{IF}\neg P)\right) = (0.5, 0.5)$,

so $P \underset{IF}{\wedge} (_{IF}\neg P)$ is not an intuitionistic fuzzy contradiction.

And if $v_N(P) = (0.5, 0.5, 0.5)$ in neutrosophic logic, also $v_N(_N\neg P) = (0.5, 0.5, 0.5)$, so $P \underset{N}{\wedge} (_N\neg P)$ is not a neutrosophic contradiction.

Many other examples can be constructed, of propositions whose degrees of their fuzzy, intuitionistic fuzzy, or neutrosophic components belong to (0,1).

A definition has to be introduced, for example in the neutrosophic logic.

A *neutrosophic contradiction threshold* should be established by the experts in respect to the

application or problem to solve: $C(t_C, i_C, f_C)$, with $t_C, i_C, f_C$ single-valued numbers in [0,1].

Then, if a proposition $P(t_P, i_P, f_P)$, is such that

$$(t_P, i_P, f_P) \leq_N (t_C, i_C, f_C), \quad (31)$$

then $P$ is a *neutrosophic contradiction*.

If $(t_P, i_P, f_P)_N > (t_C, i_C, f_C)$, then $P$ is not a neutrosophic contradiction.

While, if $(t_P, i_P, f_P)$ is neiter $\leq_N (t_C, i_C, f_C)$ nor $\geq_N (t_C, i_C, f_C)$, then $P$ is neither a neutrosophic contradiction, nor a neutrosophic non-contradiction. We talk about *neutrosophic undecidability*.

b) Another definition of completeness in classical proof theory is that in a given formal system, either every closed sentence is provable or its negation is provable.

But, again in fuzzy, intuitionistic fuzzy, and neutrosophic logic systems, we deal with *partial provability*, since an implication $A \underset{F}{\rightarrow} B$ or $A \underset{IF}{\rightarrow} B$ or respectively $A \underset{N}{\rightarrow} B$ have, in general, a partial degree of truth (provability), not a 100% truth.

Therefore, again in neutrosophic logic the experts need to establish a *neutrosophic provability threshold* $\pi(t_\pi, i_\pi, f_\pi)$, with $t_\pi, i_\pi, f_\pi \in [0,1]$.

Then: a proposition $B$ is neutrosophically provable into the system if there exists a proposition $A$, neutrosophically proven into the system, such that:

$$v_N(A \underset{N}{\rightarrow} B)_N \geq (t_\pi, i_\pi, f_\pi). \tag{32}$$

If

$$v_N(D \underset{N}{\rightarrow} B) <_N (t_\pi, i_\pi, f_\pi) \tag{33}$$

for any neutrosophic provable proposition $D$ into the system, then $B$ is *neutrosophically unproven*.

c) From a *syntactical completeness* point of view, in a classical formal system, given a closed formula $\psi$, either $\psi$ or $\neg\psi$ is a theorem of the system.

Again, we provide the same answer as at point 15.a), we mean that both $\psi$ and $\neg\psi$ may be partially true theorems in fuzzy, intuitionistic fuzzy, and neutrosophic systems.

d) *Completeness with respect to a given property*, in classical metalogic, is referred to the fact that: in a formal system, any formula $\psi$, that has the property, can be derived.

Again, we provide an answer similar to that at point 15.b), we mean that any formula $\psi$ in the

system, can be partially derived in fuzzy, intuitionistic fuzzy, and neutrosophic systems.

Sections 12-15 showed that a neutrosophic tautology ($q$) does not necessarily have the neutrosophic valuation $v_n(q) = (1, 0, 0)$, as supposed by Rivieccio, but $v_n(q)_N \geq v_n(\tau)$ or the valuation of $q$ has to be greater than or equal to the valuation of the neutrosophic tautological threshold.

Rivieccio proves that Ashbacher's Paraconsistent Neutrosophic Logic, in the particular case when $i$ = 0, $t$ = 0 or 1, and $f$ = 0 or 1, with the connectives $(N_2), (C_3),$ and $(D_1)$, yields the exact truth table of Belnap.

In general, Neutrosophic Logic based on the triplet ($t$, $i$, $f$) is more complex than Belnap's four-valued logic, while its extension, called Refined Neutrosophic Logic [6], refines each neutrosophic component:

$t$ as $t_1, t_2, \ldots, t_p$, for $p \geq 1$;

$i$ as $i_1, i_2, \ldots, i_r$, for $r \geq 1$;

and $f$ as $f_1, f_2, \ldots, f_s$, for $s \geq 1$;

while in Belnap's logic there are only two (sub)indeterminacies: contradiction, and unknown.

Neutrosophic Logic is more flexible, adjustable to each practical application, having each neutrosophic component split in as many subcomponents as needed to solve the problem.

Belnap's Logic is mostly a theoretical approach.

## *XI.2.16. Laws of Classical Logic That Do Not Hold in The Interval Neutrosophic Logic*

Rivieccio lists several laws of classical logic that do not hold in the Interval Neutrosophic Logic [8], such as: excluded middle, non-contradiction, contraposition, and Pseudo Scotus.

This should be normal, in our opinion, that when passing from a classical [logic, set, and probability] theory in this case, many classical laws, properties, rules, theorems would not work, since in classical theory one deals with full-truths ($t = 1$), while in modern theories with partial-truths ($0 < t < 1$) in general.

17) "Another rather straightforward extension would be to let *T, I, F* be subsets of some partially or linearly ordered lattice *L* instead of the real unit interval [0,1]" (Rivieccio, p. 1867).

We have extended the *T, I, F* single-valued or subset-valued neutrosophic components above 1 (one) [that we called: neutrosophic overset / overlogic / overprobability, and so on], and below 0 (zero) [that we have called: neutrosophic underset / underlogic / underprobability and so on]. See Smarandache, 2016 [9-11].

And we combined both over- and under- in order to get *T, I, F* below 0 (zero) and above 1 (one) in what we have called: neutrosophic offset / offlogic / offprobability, and so on.

Another extension of *T, I, F* was done in the frame of Complex Neutrosophic Set, as follows {see M. Ali & F. Smarandache, 2017 [12]}: (34)

$T = t_1 e^{jt_2}, I = i_1 e^{ji_2}, F = f_1 e^{jf_2},$ with $j = \sqrt{-1}$;

where the amplitudes $t_1, i_{\underline{1}}, f_1$ are subsets of [0,1], together with their corresponding phases $t_2, i_2, f_2$ as angles between $[0, 2\pi]$, are parts of the unit circle.

Even more, we extended to bipolar / tripolar / multipolar neutrosophic set, and respectively bipolar / tripolar / multipolar complex neutrosophic set {[9], pp. 144-147}.

The next step will be to consider, as Rivieccio has suggested, a partially (if possible totally) ordered neutrosophic lattice.

## *XI.2.18. Modal Contexts*

"... the possibility to deal with modal contexts" (Rivieccio, p. 1867).

We have defined several types of neutrosophic modal logic. See our paper *Neutrosophic Modal Logic*, in this book.

## *XI.2.19. Neutrosophic Score Function*

"... it would be very useful to define suitable order relations on the set of neutrosophic truth values" (Rivieccio, p. 1867).

Indeed, the neutrosophic order relationships *(N1)* and *(N2)* defined previously are partial orders, and they leave room for neutrosophic propositions that are neither neutrosophic tautologies, nor neutrosophic nontautologies,

neither neutrosophic contradictions, nor non-contradictions, and so on.

Since 2008, new neutrosophic ordering relationships have been designed, such as neutrosophic score function *(s),* neutrosophic accuracy function *(a),* and neutrosophic certainty function *(c).* Applying all three of them, one after the other in this order *(s),* then *(a),* and afterwards *(c),* we are able to get a total order relationship between the neutrosophic numbers. Yet, better ordering realtionships can be designed.

See also *The Average Positive Qualitative Neutrosophic Function and The Average Negative Qualitative Neutrosophic Function* above, pp. 9-13.

*Applications.*

They have been successfully applied in multi-criteria decision making for comparing single-valued and interval-valued neutrosophic numbers in selecting the optimum alternative.

*XI.2.20.* In search for a neutrosophic *total order* on the set of single-valued neutrosophic triplets, another idea would be to compare $(t_1, i_1, f_1)$ with $(t_2, i_2, f_2)$ by computing their

similarity measures with respect to the ideal neutrosophic number $(1,0,0)$ : the closer, the bigger is.

If $s\big((1,0,0),(t_1,i_1,f_1)\big) > s\big((1,0,0),(t_2,i_2,f_2)\big)$, then $(t_1,i_1,f_1)_N > (t_2,i_2,f_2)$ , and if their similarity measures are equal, either to consider $(t_1,i_1,f_1) = (t_2,i_2,f_2)$, or use another criterion to order them.

But, because there are many neutrosophic similarity measures (e.g. cosine, tangent, dice, and others based on the distance between triplets), the big question is: which one to use?

## *XI.2.21. Neutrosophic Lattices*

### *Theorem XI.2.21.1.*

The set of neutrosophic tautologies, *Taut*, endowed with the binary operations defined as: for any $P_1(t_1,i_1,f_1)$ and $P_1(t_2i_2f_2) \in Taut$,

$$P_1 \overset{\wedge}{_N} P_2 = Q(min\{t_1,t_2\}, max\{i_1,i_2\}, max\{f_1,f_2\}) \quad (35)$$

and $P_1 \overset{\vee}{_N} P_2 = S(max\{t_1,t_2\}, min\{i_1,i_2\}, min\{f_1,f_2\})$ (36)

forms a *neutrosophic lattice*.

*Proof:*

If $P_1 \in Taut$, then $t_1 \geq t_\tau,\ i_1 \leq i_\tau,\ f_1 \leq f_\tau$;

if $P_2 \in Taut$, then $t_2 \geq t_\tau, i_2 \leq i_\tau, f_2 \leq f_\tau$.

Then:

$$P_1 \overset{\wedge}{}_N P_2 = Q(min\{t_1, t_2\}, max\{i_1, i_2\}, max\{f_1, f_2\}), \quad (37)$$

but (38)

$min\{t_1, t_2\} \geq t_\tau, max\{i_1, i_2\} \leq i_\tau$, and $max\{f_1, f_2\} \leq f_\tau$.

Therefore $\overset{\wedge}{}_N$ is well-defined on *Taut*.

Similarly, $\overset{\vee}{}_N$ is well-defined on *Taut*, because:

$$P_1 \overset{\vee}{}_N P_2 = S(max\{t_1, t_2\}, min\{i_1, i_2\}, min\{f_1, f_2\}) \quad (39)$$

and $max\{t_1, t_2\} \geq t_\tau$, $min\{i_1, i_2\} \leq i_\tau$,

and $min\{f_1, f_2\} \leq f_\tau$.

It is easly proved that:

$$P_1 \overset{\wedge}{}_N P_1 = P_1 \quad (40)$$

because $min\{t_1, t_1\} = t_1$, $max\{i_1, i_1\} = i_1$

and $max\{f_1, f_1\} = f_1$ {symmetry}.

Similarly,

$$P_1 \overset{\vee}{}_N P_1 = P_1 \quad (41)$$

because $max\{t_1, t_1\} = t_1$, $min\{i_1, i_1\} = i_1$

and $min\{f_1, f_1\} = f_1$ {symmetry}.

Also, $\overset{\wedge}{}_N$ and $\overset{\vee}{}_N$ are associative, since:

$$P_1 \overset{\wedge}{}_N \left(P_1 \overset{\vee}{}_N P_2\right) = P_1 \overset{\vee}{}_N \left(P_1 \overset{\wedge}{}_N P_2\right) = P_1 \quad (42)$$

because

$P_1 \underset{N}{\wedge} \left(P_1 \underset{N}{\vee} P_2\right)$ has the neutrosophic valuation:

$$(t_1, i_1, f_1) \underset{N}{\wedge} (max\{t_1, t_2\}, min\{i_1, i_2\}, min\{f_1, f_2\}) =$$

$$\left(min\{t_1, max\{t_1, t_2\}\}, max\{i_1, min\{i_1, i_2\}\}, max\{f_1, min\{f_1, f_2\}\}\right) =$$

$$(t_1, i_1, f_1). \quad (43)$$

And in a similar way:

$$P_1 \underset{N}{\vee} \left(P_1 \underset{N}{\wedge} P_2\right) \quad (44)$$

has the neutrosophic valuation:

$$(t_1, i_1, f_1) \underset{N}{\vee} (min\{t_1, t_2\}, max\{i_1, i_2\}, max\{f_1, f_2\}) =$$

$$\left(max\{t_1, min\{t_1, t_2\}\}, min\{i_1, max\{i_1, i_2\}\}, min\{f_1, max\{f_1, f_2\}\}\right) =$$

$$(t_1, i_1, f_1). \quad (45)$$

The minimum element in the neutrosophic tautological lattice *Taut* is

$$\tau(t_\tau, i_\tau, f_\tau)$$

and the maximum element is $1(1, 0, 0)$.

*Theorem XI.2.21.2.*

The set of neutrosophic non-tautologies, *NonTaut*, with respect to the same binary operations $\underset{N}{\wedge}$ and $\underset{N}{\vee}$, also forms a neutrosophic

lattice, whose minimum element is $0(0,1,1)$ , and its supremum element is $\tau(t_\tau, i_\tau, f_\tau)$.

*Theorem XI.2.21.3.*

The set of all neutrosophic contradictions, endowed with the same binary operations $\overset{\wedge}{N}$ and $\overset{\vee}{N}$, is a neutrosophic lattice, whose minimum element is 0(0,1,1) and maximum element is the neutrosophic contradiction threshold $C(t_c, i_c, f_c)$.

Same style of proof.

## *Conclusion*

K.T. Atanassov, C. Cornelis and E.E. Kerre [6] said about neutrosophy the following: "these ideas, once properly formalized, will have a profound impact on our future dealings with imprecision."

Then Dr. Umberto Rivieccio concludes his paper: "We share their opinion, and hope that this paper will encourage others to pursue deeper investigations that may lead to such proper formalization".

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**CHAPTER XII**

## *XII.1. Neutrosophic Predicate Logic*

*XII.1.1. Neutrosophic Propositional Logic* deals with propositions $\mathcal{P}$ that have a degree of truth (*T*), a degree of indeterminacy (*I*), and a degree of falsehood (*F*), where in the most general case *T, I, F*, are subsets of the interval [0,1].

Particular cases have been studied so far, such as: when *T, I, F* are single values in [0,1], or *T, I, F* are interval-valued in [0,1], or *T, I, F* as discrete subsets of [0,1], and so on.

*XII.1.2. Neutrosophic Predicate Logic* (or *Neutrosophic First-Order Logic*, or *Neutrosophic Quantified Logic*) is a generalization of Neutrosophic Propositional Logic and of Classical Predicate Logic. As a neutrosophic formal language, Neutrosophical Predicate Logic deals with *neutrosophic predi-cates*, *neutrosophic variables*, and *neutrosophic quantifiers*, which are predicates / variables / and quantifiers respectively that deal with indeterminacy. It is used in

*neutrosophic expert systems* for automatic reasoning with the help of computer programs.

*XII.1.3.* Neutrosophic Predicate is a generalization of the Neutrosophic Relation.

A neutrosophic predicate with one argument is referred to as *neutrosophic monoadic*, with two arguments is referred to as *neutrosophic dyadic*, and in general with $n$ arguments, for integer $n \geq 1$, is referred to as *neutrosophic n-adic*.

The neutrosophic predicate is also a generalization a neutrosophic propositional variable, since a neutrosophic propositional variable can be treated as a neutrosophic predicate with zero arguments.

*Examples.*

Let's consider the proposition:

$\mathcal{P}$ = "John is a logician".

In classical logic, proposition $\mathcal{P}$ is either true (1), or false (0).

In neutrosophic logic (NL) proposition $\mathcal{P}$ may be partially true (let's say T = 0.4), partially false (since John also does research in other fields, such as non-Euclidean geometry and algebraic struc-

tures for example; let's say F = 0.5), and partially indeterminate (since John does in secret unknown research in another field; let's say I = 0.2).

Therefore NL($\mathcal{P}$) = (0.4, 0.2, 0.5) in neutrosophic propositional logic.

Let's extend this example to the neutrosophic predicate:

$\mathcal{P}$(X) = "X is a neutrosophic logician",

where X is a human being from our planet.

The neutrosophic truth-value of $\mathcal{P}$(X) is ($t_x$, $i_x$, $f_x$), where $t_x$, $i_x$, $f_x$ are subsets of the interval *[0,1].*

The universe of discourse is formed by all human beings from Earth.

The predicate "is a neutrosophic logician" takes one variable, "X". We can extend it to *n*-variables, $n \geq 2$:

A($X_1$, $X_2$, ..., $X_n$) = "$X_1$, and $X_2$, and ... and $X_n$ are logicians", whose neutrosophic truth-value is

$$(t_{X_1,X_2,...,X_n}, i_{X_1,X_2,...,X_n}, f_{X_1,X_2,...,X_n}) \in [0,1]\times[0,1]\times[0,1].$$

## *XII.1.4. Neutrosophic Quantifiers*

### *XII.1.4.1. Neutrosophic Existential Quantifier.*

Let $\mathcal{U}$ be the universal set, representing all faculty from the University Alpha.

$\exists x\mathcal{P}(X)$ = "There exists a faculty $x \in \mathcal{U}$ such that X is a neutrosophic logician".

But at the University Alpha there may be faculty that work part-time, full-time, or even over-time.

Thus, the neutrosophic truth-value of the variable proposition "$\exists x\mathcal{P}(X)$" may be <1, or >1, with respect to some of its neutrosophic components $t_{\exists x}, i_{\exists x}, f_{\exists f}$.

### *XII.1.4.2. Neutrosophic Universal Quantifier.*

$\forall x\mathcal{P}(x)$ = "Any faculty $x \in \mathcal{U}$ is a neutrosophic logician".

Similarly, the neutrosophic truth-value of the variable proposition "$\forall x\mathcal{P}(x)$" is $(t_{\forall x}, i_{\forall x}, f_{\forall x})$, where each component may be above 1, equal to 1, or below 1.

## *XII.2. Neutrosophic Decidability System*

An incomplete system of axioms gives birth to a *partial theory*.

But, if we introduce two contradictory axioms into an axiomatic system, we get a *contradictory system*.

A *neutrosophic axiomatic system* is a system that contains at least a $(t,i,f) \neq (1,0,0)$ axiom, meaning an axiom that is not 100% true, or at least two axioms that have a non-null degree of contradiction.

A proposition in a neutrosophic axiomatic system has some degree of decidability ($d$), some degree of indeterminate-decidability ($i$), and some degree of undecidability ($u$), i.e. it is a $(d,i,u)$-decidable proposition, where $d, i, u \subseteq [0, 1]$.

We can introduce *tresholds* for decidability ( $tres_d$ ), indeterminate-decidability ( $tres_i$ ), and undecidability ($tres_u$) respectively, or $d \geq tres_d$, $i \leq tres_i$ , and $u \leq tres_u$ respectively {when $d,i,u$ are crisp numbers in $[0, 1]$}; but if $d, i, u$ are subsets of $[0, 1]$, we may consider either *sup(d), sup(i),*

*sup(u)*, or *mid(d), mid(i), mid(u),* with *mid(.)* being the midpoint of the set, or other function-values, as $f(d), f(i), f(u)$ respectively, depending on the application, where $f: \mathcal{P}([0,1]) \to [0,1]$, and $\mathcal{P}([0,1])$ is the set of all subsets of *[0, 1]*.

# *XII.3. Neutrosophic Modal Logic*

## *Abstract.*

We introduce now for the first time the neutrosophic modal logic. The Neutrosophic Modal Logic includes the neutrosophic operators that express the modalities. It is an extension of neutrosophic predicate logic and of neutrosophic propositional logic.

## *XII.3.1. Introduction.*

The paper extends the fuzzy modal logic [1, 2, and 4], fuzzy environment [3] and neutrosophic sets, numbers and operators [5 – 12], together with the last developments of the neutrosophic environment {including *(t,i,f)*-neutrosophic algebraic structures, neutrosophic triplet structures, and neutrosophic overset / underset / offset} [13 - 15] passing through the symbolic neutrosophic logic [16], ultimately to neutrosophic modal logic.

This paper also answers Rivieccio's question on neutrosophic modalities.

All definitions, sections, and notions introduced in this paper were never done before,

neither in my previous work nor in other researchers'.

Therefore, we introduce now the Neutrosophic Modal Logic and the Refined Neutrosophic Modal Logic. Then we can extend them to Symbolic Neutrosophic Modal Logic and Refined Symbolic Neutrosophic Modal Logic, using labels instead of numerical values.

There is a large variety of neutrosophic modal logics, as actually happens in classical modal logic too. Similarly, the neutrosophic accessibility relation and possible neutrosophic worlds have many interpretations, depending on each particular application. Several neutrosophic modal applications are also listed.

Due to numerous applications of neutrosophic modal logic (see the examples throughout the paper), the introduction of the neutrosophic modal logic was needed.

*Neutrosophic Modal Logic* is a logic where some neutrosophic modalities have been included.

Let $\mathcal{P}$ be a neutrosophic proposition. We have the following types of *neutrosophic modalities*:

I. *Neutrosophic Alethic Modalities* (related to truth) has three neutrosophic operators:

*Neutrosophic Possibility*: It is neutrosophically possible that $\mathcal{P}$.

*Neutrosophic Necessity*: It is neutrosophically necessary that $\mathcal{P}$.

*Neutrosophic Impossibility*: It is neutrosophically impossible that $\mathcal{P}$.

II. *Neutrosophic Temporal Modalities* (related to time)

It was the neutrosophic case that $\mathcal{P}$.

It will neutrosophically be that $\mathcal{P}$.

And similarly:

It has always neutrosophically been that $\mathcal{P}$.

It will always neutrosophically be that $\mathcal{P}$.

III. *Neutrosophic Epistemic Modalities* (related to knowledge):

It is neutrosophically known that $\mathcal{P}$.

IV. *Neutrosophic Doxastic Modalities* (related to belief):

It is neutrosophically believed that $\mathcal{P}$.

V. *Neutrosophic Deontic Modalities*:
It is neutrosophically obligatory that $\mathcal{P}$.
It is neutrosophically permissible that $\mathcal{P}$.

## *XII.3.2. Neutrosophic Alethic Modal Operators*

The modalities used in classical (alethic) modal logic can be neutrosophicated by inserting the indeterminacy.

We insert the *degrees of possibility* and *degrees of necessity*, as refinement of classical modal operators.

## *XII.3.3. Neutrosophic Possibility Operator.*

The classical Possibility Modal Operator «$\Diamond P$» meaning «It is possible that *P*» is extended to *Neutrosophic Possibility Operator*: $\Diamond_N \mathcal{P}$ meaning «It is (*t, i, f*)-possible that $\mathcal{P}$ », using Neutrosophic Probability, where «(t, i, f)-possible» means *t* % possible (chance that $\mathcal{P}$ occurs), *i* % indeterminate (indeterminate-chance that $\mathcal{P}$ occurs), and *f* % impossible (chance that $\mathcal{P}$ does not occur).

If $\mathcal{P}(t_p, i_p, f_p)$ is a neutrosophic proposition, with $t_p, i_p, f_p$ subsets of [0, 1], then the neutrosophic

truth-value of the neutrosophic possibility operator is:

$$\Diamond_N \mathcal{P} = \left(\sup(t_p), \inf(i_p), \inf(f_p)\right), \qquad (1)$$

which means that if a proposition $P$ is $t_p$ true, $i_p$ indeterminate, and $f_p$ false, then the value of the neutrosophic possibility operator $\Diamond_N \mathcal{P}$ is: $\sup(t_p)$ possibility, $\inf(i_p)$ indeterminate-possibility, and $\inf(f_p)$ impossibility.

For example.

Let P = «It will be snowing tomorrow».

According to the meteorological center, the neutrosophic truth-value of $\mathcal{P}$ is:

$\mathcal{P}([0.5, 0.6], (0.2, 0.4), \{0.3, 0.5\})$,

i.e. $[0.5, 0.6]$ true, $(0.2, 0.4)$ indeterminate, and $\{0.3, 0.5\}$ false.

Then the neutrosophic possibility operator is:

$\Diamond_N \mathcal{P} = (\sup[0.5, 0.6], \inf(0.2, 0.4), \inf\{0.3, 0.5\}) = (0.6, 0.2, 0.3)$,

i.e. 0.6 possible, 0.2 indeterminate-possibility, and 0.3 impossible.

## *XII.3.4. Neutrosophic Necessity Operator*

The classical Necessity Modal Operator «$\Box P$» meaning «It is necessary that $P$» is extended to *Neutrosophic Necessity Operator*: $\Box_N \mathcal{P}$ meaning «It is ($t$, $i$, $f$)-necessary that $\mathcal{P}$ », using again the Neutrosophic Probability, where similarly «(t, i, f)-necessity» means $t$ % necessary (surety that $\mathcal{P}$ occurs), $i$ % indeterminate (indeterminate-surety that $\mathcal{P}$ occurs), and $f$ % unnecessary (unsurely that $\mathcal{P}$ occurs).

If $\mathcal{P}(t_p, i_p, f_p)$ is a neutrosophic proposition, with $t_p, i_p, f_p$ subsets of [0, 1], then the neutrosophic truth value of the neutrosophic necessity operator is:

$$\Box_N \mathcal{P} = \left(\inf(t_p), \sup(i_p), \sup(f_p)\right), \qquad (2)$$

which means that if a proposition $\mathcal{P}$ is $t_p$ true, $i_p$ indeterminate, and $f_p$ false, then the value of the neutrosophic necessity operator $\Box_N \mathcal{P}$ is: $\inf(t_p)$ necessary, $\sup(i_p)$ indeterminate-necessity, and $\sup(f_p)$ unnecessary.

Taking the previous *example*:

$\mathcal{P}$ = «It will be snowing tomorrow», with $\mathcal{P}([0.5, 0.6], (0.2, 0.4), \{0.3, 0.5\})$,

then the neutrosophic necessity operator is:

$\square_N\mathcal{P} = (\inf[0.5, 0.6], \sup(0.2, 0.4), \sup\{0.3, 0.5\}) = (0.5, 0.4, 0.5)$,

i.e. 0.5 necessary, 0.4 indeterminate-necessity, and 0.5 unnecessary.

## *XII.3.5. Other Possibility and Necessity Operators*

The previously defined neutrosophic possibility and respectively neutrosophic necessity operators, for $\mathcal{P}(t_p, i_p, f_p)$ a neutrosophic proposition, with $t_p, i_p, f_p$ subset-valued included in *[0, 1]*,

$\diamond_N\mathcal{P}$ = (sup($t_p$), inf($i_p$), inf($f_p$)),

$\square_N\mathcal{P}$ = (inf($t_p$), sup($i_p$), sup($f_p$)),

work quite well for subset-valued (including interval-valued) neutrosophic components, but they fail for single-valued neutrosophic components because one gets $\diamond_N\mathcal{P} = \square_N\mathcal{P}$.

Depending on the applications, more possibility and necessity operators may be defined.

Their definitions may work, mostly based on *max / min / min* for possibility operator and *min / max / max* for necessity operator ( when dealing with single-valued neutrosophic components in *[0, 1]* ), or based on *sup / inf / inf* for possibility operator and *inf / sup / sup* for necessity operator ( when dealing with interval-valued or more general with subset-valued of neutrosophic components included in *[0, 1]* ):

For examples.

Let $\mathcal{P}(t_p, i_p, f_p)$ be a neutrosophic proposition, with $t_p, i_p, f_p$ single-valued of *[0, 1]*, then the neutrosophic truth-value of the neutrosophic possibility operator is:

$\Diamond_N \mathcal{P} = ( max\{t_p, 1-f_p\}, min\{i_p, 1-i_p\}, min\{f_p, 1- t_p\} )$

or

$\Diamond_N \mathcal{P} = ( max\{t_p, 1-t_p\}, min\{i_p, 1-i_p\}, min\{f_p, 1- f_p\} )$

or

$\Diamond_N \mathcal{P} = (1- f_p, i_p, f_p)$

{defined by Anas Al-Masarwah}.

Let $\mathcal{P}(t_p, i_p, f_p)$ be a neutrosophic proposition, with $t_p, i_p, f_p$ single-valued of *[0, 1]*, then the

neutrosophic truth-value of the neutrosophic necessity operator is:

$$\Box_N \mathcal{P} = ( min\{t_p, 1\text{-}f_p\}, max\{i_p, 1\text{-}i_p\}, max\{f_p, 1\text{-}t_p\} )$$

or

$$\Box_N \mathcal{P} = ( min\{t_p, 1\text{-}t_p\}, max\{i_p, 1\text{-}i_p\}, max\{f_p, 1\text{-}f_p\} )$$

or

$$\Box_N \mathcal{P} = (t_p, i_p, 1 - t_p)$$

{defined by Anas Al-Masarwah}.

The above six defined operators may be extended from single-valued numbers of *[0, 1]* to interval and subsets of *[0, 1],* by simply replacing the subtractions of numbers by subtractions of intervals or subsets, and "*min*" by "*inf*", while "*max*" by "*sup*".

## *XII.3.6. Connection between Neutrosophic Possibility Operator and Neutrosophic Necessity Operator.*

In classical modal logic, a modal operator is equivalent to the negation of the other:

$$\Diamond P \leftrightarrow \neg\Box\neg P, \quad (3)$$

$$\Box P \leftrightarrow \neg \Diamond \neg P. \quad (4)$$

In neutrosophic logic one has a class of neutrosophic negation operators. The most used one is:

$$\overset{\neg}{}_{N}P(t,i,f) = \bar{P}(f,1-i,t), \qquad (5)$$

where t, i, f are real subsets of the interval *[0, 1].*

Let's check what's happening in the neutrosophic modal logic, using the previous *example*.

One had:

$$\mathcal{P}([0.5,0.6],(0.2,0.4),\{0.3,0.5\}),$$

then

$$\overset{\neg}{}_{N}\mathcal{P} = \bar{\mathcal{P}}(\{0.3,0.5\},1-(0.2,0.4),[0.5,0.6]) =$$

$$\bar{\mathcal{P}}(\{0.3,0.5\},1-(0.2,0.4),[0.5,0.6]) =$$

$$\bar{\mathcal{P}}(\{0.3,0.5\},(0.6,0.8),[0.5,0.6]).$$

Therefore, denoting by $\overset{\leftrightarrow}{}_{N}$ the neutrosophic equivalence, one has:

$$\overset{\neg}{}_{N}\overset{\square}{}_{N}\overset{\neg}{}_{N}\mathcal{P}([0.5,0.6],(0.2,0.4),\{0.3,0.5\})\overset{\leftrightarrow}{}_{N}$$

$\overset{\leftrightarrow}{}_{N}$ It is not neutrosophically necessary that «It will not be snowing tomorrow»

$\overset{\leftrightarrow}{}_{N}$ It is not neutrosophically necessary that $\bar{\mathcal{P}}(\{0.3,0.5\},(0.6,0.8),[0.5,0.6])$

$\overset{\leftrightarrow}{N}$ It is neutrosophically possible that $\overset{\neg}{N}\bar{\mathcal{P}}(\{0.3, 0.5\}, (0.6, 0.8), [0.5, 0.6])$

$\overset{\leftrightarrow}{N}$ It is neutrosophically possible that $\mathcal{P}([0.5, 0.6], 1 - (0.6, 0.8), \{0.3, 0.5\})$

$\overset{\leftrightarrow}{N}$ It is neutrosophically possible that $\mathcal{P}([0.5, 0.6], (0.2, 0.4), \{0.3, 0.5\})$

$\overset{\leftrightarrow}{N}\overset{\Diamond}{N}\mathcal{P}([0.5, 0.6], (0.2, 0.4), \{0.3, 0.5\}) = (0.6, 0.2, 0.3).$

Let's check the second neutrosophic equivalence.

$\overset{\neg}{N}\overset{\Diamond}{N}\overset{\neg}{N}\mathcal{P}([0.5, 0.6], (0.2, 0.4), \{0.3, 0.5\})\overset{\leftrightarrow}{N}$

$\overset{\leftrightarrow}{N}$ It is not neutrosophically possible that «It will not be snowing tomorrow»

$\overset{\leftrightarrow}{N}$ It is not neutrosophically possible that $\bar{\mathcal{P}}(\{0.3, 0.5\}, (0.6, 0.8), [0.5, 0.6])$

$\overset{\leftrightarrow}{N}$ It is neutrosophically necessary that $\overset{\neg}{N}\bar{\mathcal{P}}(\{0.3, 0.5\}, (0.6, 0.8), [0.5, 0.6])$

$\overset{\leftrightarrow}{N}$ It is neutrosophically necessary that $\mathcal{P}([0.5, 0.6], 1 - (0.6, 0.8), \{0.3, 0.5\})$

$\overset{\leftrightarrow}{N}$ It is neutrosophically necessary that $\mathcal{P}([0.5, 0.6], (0.2, 0.4), \{0.3, 0.5\})$

$\overset{\leftrightarrow}{N}\overset{\square}{N}\mathcal{P}([0.5, 0.6], (0.2, 0.4), \{0.3, 0.5\}) = (0.6, 0.2, 0.3)$.

## *XII.3.7. Neutrosophic Modal Equivalences*

Neutrosophic Modal Equivalences hold within a certain accuracy, depending on the definitions of neutrosophic possibility operator and neutrosophic necessity operator, as well as on the definition of the neutrosophic negation – employed by the experts depending on each application. Under these conditions, one may have the following neutrosophic modal equivalences:

$$\Diamond_N \mathcal{P}(t_p, i_p, f_p) \overset{\leftrightarrow}{N}\neg\overset{\square}{N}\neg\mathcal{P}(t_p, i_p, f_p) \tag{6}$$

$$\square_N \mathcal{P}(t_p, i_p, f_p) \overset{\leftrightarrow}{N}\neg\overset{\Diamond}{N}\neg\mathcal{P}(t_p, i_p, f_p) \tag{7}$$

For example, other definitions for the *neutrosophic modal operators* may be:

$$\Diamond_N \mathcal{P}(t_p, i_p, f_p) = \left(\sup(t_p), \sup(i_p), \inf(f_p)\right), \tag{8}$$

or

$$\Diamond_N \mathcal{P}(t_p, i_p, f_p) = \left(\sup(t_p), \frac{i_p}{2}, \inf(f_p)\right) \text{ etc.}, \tag{9}$$

while

$$\square_N \mathcal{P}(t_p, i_p, f_p) = \left(\inf(t_p), \inf(i_p), \sup(f_p)\right), \qquad (10)$$

or

$$\square_N \mathcal{P}(t_p, i_p, f_p) = \left(\inf(t_p), 2i_p \cap [0,1], \sup(f_p)\right) \qquad (11)$$

etc.

## *XII.3.8. Neutrosophic Truth Threshold*

In neutrosophic logic, first we have to introduce a *neutrosophic truth threshold*, $TH = \langle T_{th}, I_{th}, F_{th} \rangle$, where $T_{th}, I_{th}, F_{th}$ are subsets of [0, 1]. We use upper-case letters (T, I, F) in order to distinguish the neutrosophic components of the threshold from those of a proposition in general.

We can say that the proposition $\mathcal{P}(t_p, i_p, f_p)$ is *neutrosophically true* if:

$$\inf(t_p) \geq \inf(T_{th}) \text{ and } \sup(t_p) \geq \sup(T_{th}); \qquad (12)$$

$$\inf(i_p) \leq \inf(I_{th}) \text{ and } \sup(t_p) \leq \sup(I_{th}); \qquad (13)$$

$$\inf(f_p) \leq \inf(F_{th}) \text{ and } \sup(f_p) \leq \sup(F_{th}). \qquad (14)$$

For the particular case when all $T_{th}, I_{th}, F_{th}$ and $t_p, i_p, f_p$ are single-valued numbers from the interval [0, 1], then one has:

The proposition $\mathcal{P}(t_p, i_p, f_p)$ is *neutrosophically true* if:

$t_p \geq T_{th}$;

$i_p \leq I_{th}$;

$f_p \leq F_{th}$.

The neutrosophic truth threshold is established by experts in accordance to each application.

## *XII.3.9. Neutrosophic Semantics.*

*Neutrosophic Semantics* of the *Neutrosophic Modal Logic* is formed by a *neutrosophic frame* $G_N$, which is a *non-empty neutrosophic set*, whose elements are called *possible neutrosophic worlds*, and a *neutrosophic binary relation* $\mathcal{R}_N$, called *neutrosophic accesibility relation*, between the possible neutrosophic worlds. By notation, one has:

$\langle G_N, \mathcal{R}_N \rangle$.

A neutrosophic world $w'_N$ that is neutrosophically accessible from the neutrosophic world $w_N$ is symbolized as:

$w_N \mathcal{R}_N w'_N$.

In a *neutrosophic model* each neutrosophic proposition $\mathcal{P}$ has a *neutrosophic truth-value* $(t_{w_N}, i_{w_N}, f_{w_N})$ respectively to each *neutrosophic world* $w_N \in G_N$, where $t_{w_N}, i_{w_N}, f_{w_N}$ are subsets of [0, 1].

A *neutrosophic actual world* can be similarly noted as in classical modal logic as $w_N *$ .

*Formalization*

Let $S_N$ be a set of neutrosophic propositional variables.

## *XII.3.10. Neutrosophic Formulas.*

1. Every neutrosophic propositional variable $\mathcal{P} \in S_N$ is a neutrosophic formula.

2. If *A, B* are neutrosophic formulas, then $\overset{\neg}{}_N A$, $A \overset{\wedge}{}_N B$ , $A \overset{\vee}{}_N B$ , $A \overset{\rightarrow}{}_N B$ , $A \overset{\leftrightarrow}{}_N B$ , and $\overset{\Diamond}{}_N A$ , $\overset{\Box}{}_N A$ , are also neutrosophic formulas, where $\overset{\neg}{}_N$, $\overset{\wedge}{}_N$, $\overset{\vee}{}_N$, $\overset{\rightarrow}{}_N$, $\overset{\leftrightarrow}{}_N$, and $\overset{\Diamond}{}_N$, $\overset{\Box}{}_N$represent the neutrosophic negation, neutrosophic intersection, neutrosophic union, neutrosophic implication, neutrosophic equivalence, and neutrosophic possibility operator, neutrosophic necessity operator respectively.

## *XII.3.11. Accesibility Relation in a Neutrosophic Theory.*

Let $G_N$ be a set of neutrosophic worlds $w_N$ such that each $w_N$ chracterizes the propositions (formulas) of a given neutrosophic theory $\tau$.

We say that the neutrosophic world $w'_N$ is accesible from the neutrosophic world $w_N$, and we write: $w_N \mathcal{R}_N w'_N$ or $\mathcal{R}_N(w_N, w'_N)$, if for any proposition (formula) $\mathcal{P} \in w_N$, meaning the neutrosophic truth-value of $\mathcal{P}$ with respect to $w_N$ is

$$\mathcal{P}\left(t_p^{w_N}, i_p^{w_N}, f_p^{w_N}\right),$$

one has the neutrophic truth-value of $\mathcal{P}$ with respect to $w'_N$

$$\mathcal{P}\left(t_p^{w'_N}, i_p^{w'_N}, f_p^{w'_N}\right),$$

where

$$\inf\left(t_p^{w'_N}\right) \geq \inf\left(t_p^{w_N}\right) \text{ and } \sup\left(t_p^{w'_N}\right) \geq \sup\left(t_p^{w_N}\right); \quad (15)$$

$$\inf\left(i_p^{w'_N}\right) \leq \inf\left(i_p^{w_N}\right) \text{ and } \sup\left(i_p^{w'_N}\right) \leq \sup\left(i_p^{w_N}\right); \quad (16)$$

$$\inf\left(f_p^{w'_N}\right) \leq \inf\left(f_p^{w_N}\right) \text{ and } \sup\left(f_p^{w'_N}\right) \leq \sup\left(f_p^{w_N}\right) \quad (17)$$

(in the general case when $t_p^{w_N}, i_p^{w_N}, f_p^{w_N}$ and $t_p^{w'_N}, i_p^{w'_N}, f_p^{w'_N}$ are subsets of the interval [0, 1]).

But in the instant of $t_p^{w_N}, i_p^{w_N}, f_p^{w_N}$ and $t_p^{w'_N}, i_p^{w'_N}, f_p^{w'_N}$ as single-values in [0, 1], the above inequalities become:

$$t_p^{w'_N} \geq t_p^{w_N}, \tag{18}$$

$$i_p^{w'_N} \leq i_p^{w_N}, \tag{19}$$

$$f_p^{w'_N} \leq f_p^{w_N}. \tag{20}$$

## *XII.3.12. Applications.*

If the neutrosophic theory $\tau$ is the Neutrosophic Mereology, or Neutrosophic Gnosisology, or Neutrosophic Epistemology etc., the neutrosophic accesibility relation is defined as above.

## *XII.3.13. Neutrosophic n-ary Accesibility Relation.*

We can also extend the classical *binary* accesibility relation $\mathcal{R}$ to a *neutrosophic n-ary accesibility relation*

$\mathcal{R}_N^{(n)}$, for $n$ integer $\geq 2$.

Instead of the classical $R(w, w')$, which means that the world $w'$ is accesible from the world $w$, we generalize it to:

$\mathcal{R}_N^{(n)}\left(w_{1_N}, w_{2_N}, \ldots, w_{n_N}; w'_N\right)$,

which means that the neutrosophic world $w_N'$ is accesible from the neutrosophic worlds $w_{1_N}, w_{2_N}, \dots, w_{n_N}$ all together.

## *XII.3.14. Neutrosophic Kripke Frame.*

$k_N = \langle G_N, R_N \rangle$ is a neutrosophic Kripke frame, since:

$i. G_N$ is an arbitrary non-empty neutrosophic set of *neutrosophic worlds*, or *neutrosophic states*, or *neutrosophic situations*.

$ii. R_N \subseteq G_N \times G_N$ is a *neutrosophic accesibility relation* of the neutrosophic Kripke frame. Actually, one has a degree of accessibility, degree of indeterminacy, and a degree of non-accessibility.

## *XII.3.15. Neutrosophic (t, i, f)-Assignement.*

The Neutrosophic *(t, i, f)*-Assignement is a neutrosophic mapping

$$v_N: S_N \times G_N \to [0,1] \times [0,1] \times [0,1] \qquad (21)$$

where, for any neutrosophic proposition $\mathcal{P} \in S_N$ and for any neutrosophic world $w_N$, one defines:

$$v_N(P, w_N) = \left(t_p^{w_N}, i_p^{w_N}, f_p^{w_N}\right) \in [0,1] \times [0,1] \times [0,1] \quad (22)$$

which is the neutrosophical logical truth value of the neutrosophic proposition $\mathcal{P}$ in the neutrosophic world $w_N$.

## *XII.3.16. Neutrosophic Deducibility.*

We say that the neutrosophic formula $\mathcal{P}$ is neutrosophically deducible from the neutrosophic Kripke frame $k_N$, the neutrosophic $(t, i, f)$ – assignment $v_N$, and the neutrosophic world $w_N$, and we write as:

$$k_N, v_N, w_N \underset{N}{\models} \mathcal{P}. \tag{23}$$

Let's make the notation:

$\alpha_N(\mathcal{P};\ k_N, v_N, w_N)$

that denotes the neutrosophic logical value that the formula $\mathcal{P}$ takes with respect to the neutrosophic Kripke frame $k_N$, the neutrosophic $(t, i, f)$-assignment $v_N$, and the neutrosphic world $w_N$.

We define $\alpha_N$ by neutrosophic induction:

1. $\alpha_N(\mathcal{P};\ k_N, v_N, w_N) \underset{=}{^{def}} v_N(\mathcal{P}, w_N)$ if $\mathcal{P} \in S_N$ and $w_N \in G_N$.

2. $\alpha_N\left(\underset{N}{\neg}\mathcal{P};\ k_N, v_N, w_N\right) \underset{=}{^{def}} \underset{N}{\neg}[\alpha_N(\mathcal{P};\ k_N, v_N, w_N)]$.

3. $\alpha_N\left(\mathcal{P}\overset{\wedge}{}_N Q;\ k_N, v_N, w_N\right) \overset{def}{=} [\alpha_N(\mathcal{P};\ k_N, v_N, w_N)]\overset{\wedge}{}_N$

$$[\alpha_N(Q;\ k_N, v_N, w_N)]$$

4. $\alpha_N\left(\mathcal{P}\overset{\vee}{}_N Q;\ k_N, v_N, w_N\right) \overset{def}{=} [\alpha_N(\mathcal{P};\ k_N, v_N, w_N)]\overset{\vee}{}_N$

$$[\alpha_N(Q;\ k_N, v_N, w_N)]$$

5. $\alpha_N\left(\mathcal{P}\overset{\rightarrow}{}_N Q;\ k_N, v_N, w_N\right) \overset{def}{=} [\alpha_N(\mathcal{P};\ k_N, v_N, w_N)]\overset{\rightarrow}{}_N$

$$[\alpha_N(Q;\ k_N, v_N, w_N)]$$

6. $\alpha_N\left(\overset{\Diamond}{}_N\mathcal{P};\ k_N, v_N, w_N\right) \overset{def}{=} \langle \text{sup, inf, inf}\rangle\{\alpha_N(\mathcal{P};\ k_N, v_N, w'_N),$ $w' \in G_N$ and $w_N R_N w'_N\}$.

7. $\alpha_N\left(\overset{\Box}{}_N\mathcal{P};\ k_N, v_N, w_N\right) \overset{def}{=} \langle \text{inf, sup, sup}\rangle\{\alpha_N(\mathcal{P};\ k_N, v_N, w'_N),$ $w'_N \in G_N$ and $w_N R_N w'_N\}$.

8. $\overset{\vDash}{}_N\mathcal{P}$ if and only if $w_N *\vDash \mathcal{P}$ (a formula $\mathcal{P}$ is neutrosophically deducible if and only if $\mathcal{P}$ is neutrosophically deducible in the actual neutrosophic world).

We should remark that $\alpha_N$ has a degree of truth $\left(t_{\alpha_N}\right)$, a degree of indeterminacy $\left(i_{\alpha_N}\right)$, and a

degree of falsehood $(f_{\alpha_N})$, which are in the general case subsets of the interval [0, 1].

Applying $\langle \sup, \inf, \inf \rangle$ to $\alpha_N$ is equivalent to calculating:

$\langle \sup(t_{\alpha_N}), \inf(i_{\alpha_N}), \inf(f_{\alpha_N}) \rangle$,

and similarly

$\langle \inf, \sup, \sup \rangle \alpha_N = \langle \inf(t_{\alpha_N}), \sup(i_{\alpha_N}), \sup(f_{\alpha_N}) \rangle$.

## *XII.3.17. Refined Neutrosophic Modal Single-Valued Logic*

Using neutrosophic (*t, i, f*) - thresholds, we refine for the first time the neutrosophic modal logic as:

### *XII.3.17.1. Refined Neutrosophic Possibility Operator.*

$\overset{\Diamond_1}{N}\mathcal{P}_{(t,i,f)}$ = «It is very little possible (degree of possibility $t_1$) that $\mathcal{P}$ », corresponding to the threshold $(t_1, i_1, f_1)$, i.e. $0 \leq t \leq t_1$, $i \geq i_1$, $f \geq f_1$, for $t_1$ a very little number in [0, 1];

$\overset{\Diamond_2}{N}\mathcal{P}_{(t,i,f)}$ = «It is little possible (degree of possibility $t_2$) that $\mathcal{P}$», corresponding to the threshold $(t_2, i_2, f_2)$, i.e. $t_1 < t \leq t_2$, $i \geq i_2 > i_1$, $f \geq f_2 > f_1$;

... ... ...

and so on;

$\underset{N}{\Diamond_m}\mathcal{P}_{(t,i,f)}$ = «It is possible (with a degree of possibility $t_m$) that $\mathcal{P}$ », corresponding to the threshold $(t_m, i_m, f_m)$, i.e. $t_{m-1} < t \leq t_m$, $i \geq i_m > i_{m-1}$, $f \geq f_m > f_{m-1}$.

*XII.3.17.2. Refined Neutrosophic Necessity Operator.*

$\underset{N}{\Box_1}\mathcal{P}_{(t,i,f)}$ = «It is a small necessity (degree of necessity $t_{m+1}$) that $\mathcal{P}$», i.e. $t_m < t \leq t_{m+1}$, $i \geq i_{m+1} \geq i_m$, $f \geq f_{m+1} > f_m$;

$\underset{N}{\Box_2}\mathcal{P}_{(t,i,f)}$ = «It is a little bigger necessity (degree of necessity $t_{m+2}$) that $\mathcal{P}$», i.e. $t_{m+1} < t \leq t_{m+2}$, $i \geq i_{m+2} > i_{m+1}$, $f \geq f_{m+2} > f_{m+1}$;

... ... ...

and so on;

$\underset{N}{\Box_k}\mathcal{P}_{(t,i,f)}$ = «It is a very high necessity (degree of necessity $t_{m+k}$) that $\mathcal{P}$», i.e. $t_{m+k-1} < t \leq t_{m+k} = 1$, $i \geq i_{m+k} > i_{m+k-1}$, $f \geq f_{m+k} > f_{m+k-1}$.

## *XII.3.18. Application of the Neutrosophic Threshold.*

We have introduced the term of (*t, i, f*)-physical law, meaning that a physical law has a degree of truth (*t*), a degree of indeterminacy (*i*), and a degree of falsehood (*f*). A physical law is 100% true, 0% indeterminate, and 0% false in perfect (ideal) conditions only, maybe in laboratory.

But our actual world ($w_N *$) is not perfect and not steady, but continously changing, varying, fluctuating.

For example, there are physicists that have proved a universal constant ($c$) is not quite universal (i.e. there are special conditions where it does not apply, or its value varies between $(c - \varepsilon, c + \varepsilon)$, for $\varepsilon > 0$ that can be a tiny or even a bigger number).

Thus, we can say that a proposition $\mathcal{P}$ is *neutrosophically nomological necessary*, if $\mathcal{P}$ is neutrosophically true at all possible neutrosophic worlds that obey the (*t, i, f*)-physical laws of the actual neutrosophic world $w_N *$.

In other words, at each possible neutrosophic world $w_N$, neutrosophically accesible from $w_N*$, one has:

$$\mathcal{P}\left(t_p^{w_N}, i_p^{w_N}, f_p^{w_N}\right) \geq TH(T_{th}, I_{th}, F_{th}), \quad (24)$$

i.e. $t_p^{w_N} \geq T_{th}$, $i_p^{w_N} \leq I_{th}$, and $f_p^{w_N} \geq F_{th}$. (25)

## *XII.3.19. Neutrosophic Mereology*

*Neutrosophic Mereology* means the theory of the neutrosophic relations among the parts of a whole, and the neutrosophic relations between the parts and the whole.

A neutrosophic relation between two parts, and similarly a neutrosophic relation between a part and the whole, has a degree of connectibility (*t*), a degree of indeterminacy (*i*), and a degree of disconnectibility (*f*).

## *XII.3.20. Neutrosophic Mereological Threshold*

Neutrosophic Mereological Threshold is defined as:

$$TH_M = (\min(t_M), \max(i_M), \max(f_M)) \quad (26)$$

where $t_M$ is the set of all degrees of con-nectibility between the parts, and between the parts and the whole;

$i_M$ is the set of all degrees of indeterminacy between the parts, and between the parts and the whole;

$f_M$ is the set of all degrees of disconnectibility between the parts, and between the parts and the whole.

We have considered all degrees as single-valued numbers.

## *XII.3.21. Neutrosophic Gnosisology*

*Neutrosophic Gnosisology* is the theory of (*t, i, f*)-knowledge, because in many cases we are not able to completely (100%) find whole knowledge, but only a part of it (*t* %), another part remaining unknown (*f* %), and a third part indeterminate (unclear, vague, contradictory) (*i* %), where *t, i, f* are subsets of the interval *[0, 1].*

## *XII.3.22. Neutrosophic Gnosisological Threshold*

Neutrosophic Gnosisological Threshold is defined, similarly, as:

$$TH_G = (\min(t_G), \max(i_G), \max(f_G)) \quad (27)$$

where $t_G$ is the set of all degrees of knowledge of all theories, ideas, propositions etc.,

$i_G$ is the set of all degrees of indeterminate-knowledge of all theories, ideas, propositions etc.,

$f_G$ is the set of all degrees of non-knowledge of all theories, ideas, propositions etc.

We have considered all degrees as single-valued numbers.

## *XII.3.23. Neutrosophic Epistemology*

And *Neutrosophic Epistemology*, as part of the Neutrosophic Gnosisology, is the theory of (*t, i, f*)-scientific knowledge. Science is infinite. We know only a small part of it (*t*%), another big part is yet to be discovered (*f*%), and a third part indeterminate (unclear, vague, contradictory) (*i*%). Of course, *t, i, f* are subsets of [0, 1].

## *XII.3.24. Neutrosophic Epistemological Threshold*

*Neutrosophic Epistemological Threshold* is defined as:

$$TH_E = (\min(t_E), \max(i_E), \max(f_E)) \qquad (28)$$

where $t_E$ is the set of all degrees of scientific knowledge of all scientific theories, ideas, propositions etc.,

$i_E$ is the set of all degrees of indeterminate scientific knowledge of all scientific theories, ideas, propositions etc.,

$f_E$ is the set of all degrees of non-scientific knowledge of all scientific theories, ideas, propositions etc.

We have considered all degrees as single-valued numbers.

## *XII.3.25. Conclusions.*

We have introduced for the first time the Neutrosophic Modal Logic and the Refined Neutrosophic Modal Logic.

*Symbolic Neutrosophic Logic* can be connected to the neutrosophic modal logic too, where

instead of numbers we may use labels, or instead of quantitative neutrosophic logic we may have a quantitative neutrosophic logic. As an extension, we may introduce *Symbolic Neutrosophic Modal Logic* and *Refined Symbolic Neutrosophic Modal Logic*, where the symbolic neutrosophic modal operators (and the symbolic neutrosophic accessibility relation) have qualitative values (labels) instead on numerical values (subsets of the interval [0, 1]).

Applications of neutrosophic modal logic are to neutrosophic modal metaphysics. Similarly to classical modal logic, there is a plethora of neutrosophic modal logics. Neutrosophic modal logics is governed by a set of neutrosophic axioms and neutrosophic rules. The neutrosophic accessibility relation has various interpretations, depending on the applications. Similarly, the notion of possible neutrosophic worlds has many interpretations, as part of possible neutrosophic semantics.

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# *XII.4. Neutrosophic Hedge Algebras*

## *Abstract*

We introduce now for the first time the neutrosophic hedge algebras as an extension of classical hedge algebras, together with an application of neutrosophic hedge algebras.

## *XII.4.1. Introduction*

The classical hedge algebras deal with linguistic variables. In neutrosophic environment we have introduced the neutrosophic linguistic variables. We have defined neutrosophic partial relationships between single-valued neutrosophic numbers. Neutrosophic operations are used in order to aggregate the neutrosophic linguistic values.

## *XII.4.2. Materials and Methods*

We introduce now, for the first time, the *Neutrosophic Hedge Algebras*, as extension of classical Hedge Algebras.

Let's consider a Linguistic Variable:

with $Dom(x)$ as the word domain of $x$, whose each element is a word (label), or string of words.

Let $\mathcal{A}$ be an attribute that describes the value of each element $x \in Dom(x)$, as follows:

$$\mathcal{A}: Dom(x) \to [0,1]^3. \quad (1)$$

$\mathcal{A}(x)$ is the neutrosophic value of $x$ with respect to this attribute:

$$A(x) = \langle t_x, i_x, f_x \rangle, \quad (2)$$

where $t_x, i_x, f_x \in [0,1]$, such that

- $t_x$ means the degree of value of $x$;
- $i_x$ means the indeterminate degree of value of $x$;
- $f_x$ means the degree of non-value of $x$.

We may also use the notation: $x\langle t_x, i_x, f_x \rangle$.

A neutrosophic partial relationship $\leq_N$ on $Dom(x)$, defined as follows:

$$x\langle t_x, i_x, f_x \rangle \leq_N y\langle t_y, i_y, f_y \rangle, \quad (3)$$

if and only if $t_x \leq t_y$, and $i_x \geq i_y$, $f_x \geq f_y$.

Therefore, $(Dom(x), \leq_N)$ becomes a neutrosophic partial order set (or neutrosophic poset), and $\leq_N$ is called a neutrosophic inequality.

Let $C = \{0, w, 1\}$ be a set of constants, $C \subset Dom(x)$, where:

- 0 = the least element, or $0_{\langle 0,1,1\rangle}$;
- w = the neutral (middle) element, or $w_{\langle 0.5,0.5,0.5\rangle}$;
- and 1 = the greatest element, or $1_{\langle 1,0,0\rangle}$.

Let $G$ be a word-set of two *neutrosophic generators*, $G \subset Dom(x)$, qualitatively a negative primary neutrosophic term (denoted $g^-$), and the other one that is qualitatively a positive primary neutrosophic term (denoted $g^+$), such that:

$$0 \leq_N g^- \leq_N w \leq_N g^+ \leq_N 1, \qquad (4)$$

or transcribed using the neutrosophic components:

$$0_{\langle 0,1,1\rangle} \leq_N g^-{}_{\langle t_{g^-}, i_{g^-}, f_{g^-}\rangle} \leq_N w_{\langle 0.5,0.5,0.5\rangle}$$

$$\leq_N g^+{}_{\langle t_{g^+}, i_{g^+}, f_{g^+}\rangle} \leq_N 1_{\langle 1,0,0\rangle},$$

where

- $0 \leq t_{g^-} \leq 0.5 \leq t_{g^+} \leq 1$ (here there are classical inequalities)
- $1 \geq i_{g^-} \geq 0.5 \geq i_{g^+} \geq 0$, and
- $1 \geq f_{g^-} \geq 0.5 \geq f_{g^+} \geq 0$.

Let $H \subset Dom(x)$ be the set of *neutrosophic hedges*, regarded as unary operations. Each hedge $h \in H$ is a functor, or comparative particle for

adjectives and adverbs as in the natural language (English).

$h: Dom(x) \to Dom(x)$

$x \to h(x)$. (5)

Instead of $h(x)$ one easily writes $\underline{hx}$ to be closer to the natural language.

By associating the neutrosophic components, one has:

$h_{\langle t_h, i_h, f_h \rangle} x_{\langle t_x, i_x, f_x \rangle}$.

A hedge applied to $x$ may increase, decrease, or approximate the neutrosophic value of the element $x$.

There also exists a *neutrosophic identity* $I \in Dom(x)$, denoted $I_{\langle 0,0,0 \rangle}$ that does not hange on the elements:

$I_{\langle 0,0,0 \rangle} x_{\langle t_x, i_x, f_x \rangle}$.

In most cases, if a hedge increases / decreases the neutrosophic value of an element $x$ situated above the neutral element $w$, the same hedge does the opposite, decreases / increases the neutrosophic value of an element $y$ situated below the neutral element $w$.

And reciprocally.

If a hedge approximates the neutrosophic value, by diminishing it, of an element $x$ situated above the neutral element $w$, then it approximates the neutrosophic value, by enlarging it, of an element $y$ situated below the neutral element $w$.

Let's refer the hedges with respect to the upper part ($\sqcup$), above the neutral element, since for the lower part ($L$) it will automatically be the opposite effect.

We split de set of hedges into three disjoint subsets:

- $H_{\sqcup}^{+}$ = the hedges that increase the neutrosophic value of the upper elements;
- $H_{\sqcup}^{-}$ = the hedges that decrease the neutrosophic value of the upper elements;
- $H_{\sqcup}^{\sim}$ = the hedges that approximate the neutrosophic value of the upper elements.

*Notations*: Let $x = x_{\sqcup} \cup w \cup x_L$ , where $x_{\sqcup}$ constitutes the upper element set, while $x_L$ the lower element subset, $w$ the neutral element. $x_{\sqcup}$ and $x_L$ are disjoint two by two.

## *XII.4.3. Operations on Neutrosophic Components*

Let $\langle t_1, i_1, f_1\rangle$, $\langle t_2, i_2, f_2\rangle$ neutrosophic numbers.

Then:

$$t_1 + t_2 = \begin{cases} t_1 + t_2, \text{if } t_1 + t_2 \leq 1; \\ 1, \text{if } t_1 + t_2 > 1; \end{cases} \tag{6}$$

and

$$t_1 - t_2 = \begin{cases} 0, \text{if } t_1 - t_2 < 0; \\ t_1 - t_2, \text{if } t_1 - t_2 \geq 0. \end{cases} \tag{7}$$

Similarly for $i_1$ and $f_1$:

$$i_1 + i_2 = \begin{cases} i_1 + i_2, \text{if } i_1 + i_2 \leq 1; \\ 1, \text{if } i_1 + i_2 > 1; \end{cases} \tag{8}$$

$$i_1 - i_2 = \begin{cases} 0, \text{if } i_1 - i_2 < 0; \\ i_1 - i_2, \text{if } i_1 - i_2 \geq 0. \end{cases} \tag{9}$$

and

$$f_1 + f_2 = \begin{cases} f_1 + f_2, \text{if } f_1 + f_2 \leq 1; \\ 1, \text{if } f_1 + f_2 > 1; \end{cases} \tag{10}$$

$$f_1 - f_2 = \begin{cases} 0, \text{if } f_1 - f_2 < 0; \\ f_1 - f_2, \text{if } f_1 - f_2 \geq 0. \end{cases} \tag{11}$$

## *XII.4.4. Neutrosophic Hedge-Element Operators*

We define the following operators:

### *XII.4.4.1. Neutrosophic Increment*

Hedge $\boxed{\uparrow}$ Element $= \langle t_1, i_1, f_1\rangle \boxed{\uparrow} \langle t_2, i_2, f_2\rangle = \langle t_2 + t_1, i_2 - i_1, f_2 - f_1\rangle$, (12)

meaning that the first triplet increases the second.

*XII.4.4.2. Neutrosophic Decrement*

Hedge $\boxed{\downarrow}$ Element $= \langle t_1, i_1, f_1\rangle \boxed{\downarrow} \langle t_2, i_2, f_2\rangle = \langle t_2 - t_1, i_2 + i_1, f_2 + f_1\rangle$, (13)

meaning that the first triplet decreases the second.

*XII.4.4.3. Theorem 1*

The neutrosophic increment and decrement operators are non-commutattive.

## *XII.4.5. Neutrosophic Hedge-Hedge Operators*

Hedge $\boxed{\uparrow}$ Hedge $= \langle t_1, i_1, f_1\rangle \boxed{\uparrow} \langle t_2, i_2, f_2\rangle = \langle t_1 + t_2, i_1 + i_2, f_1 + f_2\rangle$ (14)

Hedge $\boxed{\downarrow}$ Hedge $= \langle t_1, i_1, f_1\rangle \boxed{\downarrow} \langle t_2, i_2, f_2\rangle = \langle t_1 - t_2, i_1 - i_2, f_1 - f_2\rangle$ (15)

## *XII.4.6. Neutrosophic Hedge Operators*

Let $x_{\sqcup}\langle t_{x_{\sqcup}}, i_{x_{\sqcup}}, f_{x_{\sqcup}}\rangle \in Dom(x)$ i.e. $x_{\sqcup}$ is an upper element of $Dom(x)$, and

- $h^{+}_{\sqcup}\langle t_{h^{+}_{\sqcup}}, i_{h^{+}_{\sqcup}}, f_{h^{+}_{\sqcup}}\rangle \in H^{+}_{\sqcup}$,
- $h^{-}_{\sqcup}\langle t_{h^{-}_{\sqcup}}, i_{h^{-}_{\sqcup}}, f_{h^{-}_{\sqcup}}\rangle \in H^{-}_{\sqcup}$,
- $h^{\wedge}_{\sqcup}\langle t_{h^{\wedge}_{\sqcup}}, i_{h^{\wedge}_{\sqcup}}, f_{h^{\wedge}_{\sqcup}}\rangle \in H^{\wedge}_{\sqcup}$,

then $h^{+}_{\sqcup}$ applied to $x_{\sqcup}$ gives

$(h_{\sqcup}^{+}x_{\sqcup})\langle t_{x_{\sqcup}}, i_{x_{\sqcup}}, f_{x_{\sqcup}}\rangle \boxed{\uparrow} \langle t_{h_{\sqcup}^{+}}, i_{h_{\sqcup}^{+}}, f_{h_{\sqcup}^{+}}\rangle,$

and $h_{\sqcup}^{-}$ applied to $x_{\sqcup}$ gives

$(h_{\sqcup}^{-}x_{\sqcup})\langle t_{x_{\sqcup}}, i_{x_{\sqcup}}, f_{x_{\sqcup}}\rangle \boxed{\downarrow} \langle t_{h_{\sqcup}^{-}}, i_{h_{\sqcup}^{-}}, f_{h_{\sqcup}^{-}}\rangle,$

and $h_{\sqcup}^{\sim}$ applied to $x_{\sqcup}$ gives

$(h_{\sqcup}^{\sim}x_{\sqcup})\langle t_{x_{\sqcup}}, i_{x_{\sqcup}}, f_{x_{\sqcup}}\rangle \boxed{\downarrow} \langle t_{h_{\sqcup}^{\sim}}, i_{h_{\sqcup}^{\sim}}, f_{h_{\sqcup}^{\sim}}\rangle.$

Now, let $x_L\langle t_{x_L}, i_{x_L}, f_{x_L}\rangle \in Dom(x_L)$, i.e. $x_L$ is a lower element of $Dom(x)$. Then, $h_{\sqcup}^{+}$ applied to $x_L$ gives:

$h_{\sqcup}^{+}x_L\langle t_{x_L}, i_{x_L}, f_{x_L}\rangle \boxed{\downarrow} \langle t_{h_{\sqcup}^{+}}, i_{h_{\sqcup}^{+}}, f_{h_{\sqcup}^{+}}\rangle,$

and $h_{\sqcup}^{-}$ applied to $x_L$ gives:

$h_{\sqcup}^{-}x_L\langle t_{x_L}, i_{x_L}, f_{x_L}\rangle \boxed{\uparrow} \langle t_{h_{\sqcup}^{-}}, i_{h_{\sqcup}^{-}}, f_{h_{\sqcup}^{-}}\rangle,$

and $h_{\sqcup}^{\sim}$ applied to $x_L$ gives:

$h_{\sqcup}^{\sim}x_L\langle t_{x_L}, i_{x_L}, f_{x_L}\rangle \boxed{\uparrow} \langle t_{h_{\sqcup}^{\sim}}, i_{h_{\sqcup}^{\sim}}, f_{h_{\sqcup}^{\sim}}\rangle.$

In the same way, we may apply many increasing, decreasing, approximate or other type of hedges to the same upper or lower element

$h_{\sqcup_n}^{+}h_{\sqcup_{n-1}}^{-}h_{\sqcup}^{v} \ldots h_{\sqcup_1}^{+}x,$

generating new elements in $Dom(x)$.

The hedges may be applied to the constants as well.

*XII.4.6.1. Theorem 2*

A hedge applied to another hedge wekeans or stengthens or approximates it.

### *XII.4.6.2. Theorem 3*

If $h_{\sqcup}^{+} \in H_{\sqcup}^{+}$ and $x_{\sqcup} \in Dom(x_{\sqcup})$, then $h_{\sqcup}^{+} x_{\sqcup} \geq x_{\sqcup}$.

If $h_{\sqcup}^{-} \in H_{\sqcup}^{-}$ and $x_{\sqcup} \in Dom(x_{\sqcup})$, then $h_{\sqcup}^{-} x_{\sqcup} \geq x_{\sqcup}$.

If $h_{\sqcup}^{+} \in H_{\sqcup}^{+}$ and $x_{L} \in Dom(x_{L})$, then $h_{\sqcup}^{+} x_{L} \leq_{N} x_{L}$.

If $h_{\sqcup}^{-} \in H_{\sqcup}^{-}$ and $x_{L} \in Dom(x_{L})$, then $h_{\sqcup}^{-} x_{L} \geq_{N} x_{L}$.

### *XII.4.6.3. Converse Hedges*

Two hedges $h_1$ and $h_2 \in H$ are converse to each other, if $\forall x \in Dom(x)$, $h_1 x \leq_N x$ is equivalent to $h_2 x \geq_N x$.

### *XII.4.6.4. Compatible Hedges*

Two hedges $h_1$ and $h_2 \in H$ are compatible, if $\forall x \in Dom(x)$, $h_1 x \leq_N x$ is equivalent to $h_2 x \leq_N x$.

### *XII.4.6.5. Commutative Hedges*

Two hedges $h_1$ and $h_2 \in H$ are commutative, if $\forall x \in Dom(x)$, $h_1 h_2 x = h_2 h_1 x$. Otherwise they are called non-commutative.

### *XII.4.6.6. Cumulative Hedges*

If $h_{1_{\sqcup}}^{+}$ and $h_{2_{\sqcup}}^{+} \in H^{+}$, then two neutrosophic edges can be cumulated into one:

$$h_{1_{\sqcup}}^{+} \langle t_{h_{1_{\sqcup}}^{+}}, i_{h_{1_{\sqcup}}^{+}}, f_{h_{1_{\sqcup}}^{+}} \rangle \, h_{2_{\sqcup}}^{+} \langle t_{h_{2_{\sqcup}}^{+}}, i_{h_{2_{\sqcup}}^{+}}, f_{h_{2_{\sqcup}}^{+}} \rangle = h_{12_{\sqcup}}^{+} \langle t_{h_{1_{\sqcup}}^{+}}, i_{h_{1_{\sqcup}}^{+}}, f_{h_{1_{\sqcup}}^{+}} \rangle \boxed{\uparrow} \langle t_{h_{2_{\sqcup}}^{+}}, i_{h_{2_{\sqcup}}^{+}}, f_{h_{2_{\sqcup}}^{+}} \rangle. \quad (16)$$

Similarly, if $h_{1_\sqcup}^-$ and $h_{2_\sqcup}^- \in H^-$, then we can cumulate them into one:

$$h_{1_\sqcup}^- \langle t_{h_{1_\sqcup}^-}, i_{h_{1_\sqcup}^-}, f_{h_{1_\sqcup}^-} \rangle \, h_{2_\sqcup}^- \langle t_{h_{2_\sqcup}^-}, i_{h_{2_\sqcup}^-}, f_{h_{2_\sqcup}^-} \rangle =$$
$$h_{12_\sqcup}^- \langle t_{h_{1_\sqcup}^-}, i_{h_{1_\sqcup}^-}, f_{h_{1_\sqcup}^-} \rangle \boxed{\uparrow} \langle t_{h_{2_\sqcup}^-}, i_{h_{2_\sqcup}^-}, f_{h_{2_\sqcup}^-} \rangle. \quad (17)$$

Now, if the two hedges are converse, $h_{1_\sqcup}^+$ and $h_{1_\sqcup}^-$, but the neutrosophic components of the first (which is actually a neutrosophic number) are greater than the second, we cumulate them into one as follows: (18)

$$h_{3_\sqcup}^+ = \left(h_{1_\sqcup}^+ h_{2_\sqcup}^-\right) \langle t_{h_{1_\sqcup}^+}, i_{h_{1_\sqcup}^+}, f_{h_{1_\sqcup}^+} \rangle \boxed{\downarrow} \langle t_{h_{2_\sqcup}^-}, i_{h_{2_\sqcup}^-}, f_{h_{2_\sqcup}^-} \rangle.$$

But, if the neutrosophic components of the second are greater, and the hedges are commutative, we cumulate them into one as follows:

$$h_{3_\sqcup}^+ = \left(h_{1_\sqcup}^+ h_{2_\sqcup}^-\right) \langle t_{h_{2_\sqcup}^-}, i_{h_{2_\sqcup}^-}, f_{h_{2_\sqcup}^-} \rangle \boxed{\downarrow} \langle t_{h_{1_\sqcup}^+}, i_{h_{1_\sqcup}^+}, f_{h_{1_\sqcup}^+} \rangle \quad (19)$$

## *XII.4.7. Neutrosophic Hedge Algebra*

$NHA = (x, G, C, H \cup I, \leq_N)$ constitutes an abstract algebra, called Neutrosophic Hedge Algebra.

### *XII.4.7.1. Example of a Neutrosophic Hedge Algebra $\tau$*

Let $G = \{Small, Big\}$ the set of generators, represented as neutrosophic generators as follows:

$Small_{\langle 0.3,0.6,0.7\rangle}, Big_{\langle 0.7,0.2,0.3\rangle}$.

Let $H = \{Very, Less\}$ the set of hedges, represented as neutrosophic hedges as follows:

$Very_{\langle 0.1,0.1,0.1\rangle}, Less_{\langle 0.1,0.2,0.3\rangle}$,

where $Very \in H_{\sqcup}^{+}$ and $Less \in H_{\sqcup}^{-}$.

$x$ is a neutrosophic linguistic variable whose domain is $G$ at the beginning, but extended by generators.

The neutrosophic constants are

$C = \{0_{\langle 0,1,1\rangle}, Medium_{\langle 0.5,0.5,0.5\rangle}, 1_{\langle 1,0,0\rangle}\}$.

The neutrosophic identity is $I_{\langle 0,0,0\rangle}$.

We use the neutrosophic inequality $\leq_N$, and the neutrosophic increment / decrement operators previously defined.

Let's apply the neutrosophic hedges in order to generate new neutrosophic elements of the neutrosophic linguistic variable $x$.

$Very$ applied to $Big$ [upper element] has a positive effect:

$Very_{\langle 0.1,0.1,0.1\rangle} Big_{\langle 0.7,0.2,0.3\rangle} =$

$(Very\ Big)_{\langle 0.7+0.1,0.2-0.1,0.3-0.1\rangle} = (Very\ Big)_{\langle 0.8,0.1,0.2\rangle}$.

Then:

$Very_{\langle 0.1,0.1,0.1\rangle}(Very\ Big)_{\langle 0.9,0.1,0.2\rangle} =$

$(Very\ Very\ Big)_{\langle 0.9,0,0.1\rangle}$.

*Very* applied to *Small* [lower element] has a negative effect:

$Very_{\langle 0.1,0.1,0.1\rangle}Small_{\langle 0.3,0.6,0.7\rangle} =$

$(Very\ Small)_{\langle 0.3-0.1,0.6+0.1,0.7+0.1\rangle} =$

$(Very\ Small)_{\langle 0.2,0.7,0.8\rangle}$.

If we compute $(Very\ Very)$ first, which is a neutrosophic hedge-hedge operator:

$Very_{\langle 0.1,0.1,0.1\rangle}Very_{\langle 0.1,0.1,0.1\rangle} =$

$(Very\ Very)_{\langle 0.1+0.1,0.1+0.1,0.1+0.1\rangle} = (Very\ Very)_{\langle 0.2,0.2,0.2\rangle}$,

and we apply it to Big, we get:

$$(Very\ Very)_{\langle 0.2,0.2,0.2\rangle}Big_{\langle 0.7,0.2,0.3\rangle}$$
$$= (Very\ Very\ Big)_{\langle 0.7+0.2,0.2-0.2,0.3-0.2\rangle}$$
$$= (Very\ Very\ Big)_{\langle 0.9,0,0.1\rangle},$$

so, we get the same result.

*Less* applied to *Big* has a negative effect:

$Less_{\langle 0.1,0.2,0.3\rangle}Big_{\langle 0.7,0.2,0.3\rangle} =$

$(Less\ Big)_{\langle 0.7-0.1,0.2+0.2,0.3\rangle} = (Less\ Big)_{\langle 0.6,0.4,0.6\rangle}$.

*Less* applied to *Small* has a positive effect:

$Less_{\langle 0.1,0.2,0.3\rangle}Small_{\langle 0.3,0.6,0.7\rangle} =$

$(Less\ Small)_{\langle 0.1+0.3,0.6-0.2,0.7-0.3\rangle} = (Less\ Small)_{\langle 0.4,0.4,0.4\rangle}$.

The set of neutrosophic hedges $H$ is enriched through the generation of new neutrosophic hedges by combining a hedge with another one using the neutrosophic hedge-hedge operators.

Further, the newly generated neutrosophic hedges are applied to the elements of the linguistic variable, and more new elements are generated.

Let's compute more neutrosophic elements:

$$VLB = Very_{\langle 0.1,0.1,0.1\rangle} Less_{\langle 0.1,0.2,0.3\rangle} Big_{\langle 0.7,0.2,0.3\rangle}$$
$$= (Very\ Less\ Big)_{\left[\langle 0.1,0.1,0.1\rangle \underset{h}{\boxed{\uparrow}} \langle 0.1,0.2,0.3\rangle\right] \boxed{\downarrow} \langle 0.7,0.2,0.3\rangle}$$
$$= (Very\ Less\ Big)_{\langle 0.1+0.1,0.1+0.2,0.1+0.3\rangle \boxed{\downarrow} \langle 0.7,0.2,0.3\rangle}$$
$$= (Very\ Less\ Big)_{\langle 0.7-0.2,0.2-0.3,0.3-0.4\rangle}$$
$$= (Very\ Less\ Big)_{\langle 0.5,0,0\rangle}$$

$$VM = Very_{\langle 0.1,0.1,0.1\rangle} Medium_{\langle 0.5,0.5,0.5\rangle}$$
$$= (Very\ Medium)_{\langle 0.1,0.1,0.1\rangle \boxed{\uparrow} \langle 0.5,0.5,0.5\rangle}$$
$$= (Very\ Medium)_{\langle 0.6,0.4,0.4\rangle}$$

$$LM = Less_{\langle 0.1,0.2,0.3\rangle} Medium_{\langle 0.5,0.5,0.5\rangle}$$
$$= (Less\ Medium)_{\langle 0.1,0.2,0.3\rangle \boxed{\downarrow} \langle 0.5,0.5,0.5\rangle}$$
$$= (Less\ Medium)_{\langle 0.4,0.7,0.8\rangle}$$

$$VVS = Very_{\langle 0.1,0.1,0.1\rangle} Very_{\langle 0.1,0.1,0.1\rangle} Small_{\langle 0.3,0.6,0.7\rangle}$$
$$= (Very\ Very)_{\langle 0.2,0.2,0.2\rangle} Small_{\langle 0.3,0.6,0.7\rangle}$$
$$= (Very\ Very\ Small)_{\langle 0.1,0.8,0.9\rangle}$$

$$VLS = Very_{\langle 0.1,0.1,0.1\rangle} Less_{\langle 0.1,0.2,0.3\rangle} Small_{\langle 0.3,0.6,0.7\rangle}$$
$$= Very_{\langle 0.1,0.1,0.1\rangle} (Less\ Small)_{\langle 0.4,0.4,0.4\rangle}$$
$$= (Very\ Less\ Small)_{\langle 0.5,0.3,0.3\rangle}$$
$$LAMax = Less_{\langle 0.1,0.2,0.3\rangle} Absolute\ Maximum_{\langle 1,0,0\rangle}$$
$$= (Less\ Absolute\ Maximum)_{\langle 0.1,0.2,0.3\rangle} \boxed{\downarrow} \langle 1,0,0\rangle$$
$$= (Less\ Absolute\ Maximum)_{\langle 0.9,0.2,0.3\rangle}$$
$$LAMin = Less_{\langle 0.1,0.2,0.3\rangle} Absolute\ Minimum_{\langle 0,1,1\rangle}$$
$$= (Less\ Absolute\ Minimum)_{\langle 0.1,0.2,0.3\rangle} \boxed{\uparrow} \langle 0,1,1\rangle$$
$$= (Less\ Absolute\ Maximum)_{\langle 0.1,0.8,0.7\rangle}$$

### *XII.4.7.2. Theorem 4*

Any increasing hedge $h_{\langle t,i,f\rangle}$ applied to the absolute maximum cannot overpass the absolute maximum.

*Proof:*

$$h_{\langle t,i,f\rangle} \boxed{\uparrow} 1_{\langle 1,0,0\rangle} = (h1)_{\langle 1+t,0-i,0-f\rangle}$$
$$= (h1)_{\langle 1,0,0\rangle} = 1_{\langle 1,0,0\rangle}.$$

### *XII.4.7.3. Theorem 5*

Any decreasing hedge $h_{\langle t,i,f\rangle}$ applied to the absolute minimum cannot pass below the absolute minimum.

*Proof:*

$$h_{\langle t,i,f\rangle} \boxed{\downarrow} 0_{\langle 0,1,1\rangle} = (ho)_{\langle 0-t,1+i,1+f\rangle}$$

$= (ho)_{\langle 0,1,1\rangle} = 0_{\langle 0,1,1\rangle}$.

## *XII.4.8. Diagram of the Neutrosophic Hedge Algebra $\tau$*

| | |
|---|---|
| $1_{\langle 1,0,0\rangle}$ | ABSOLUTE MAXIMUM |
| $VVB_{\langle 0.9,0,0.1\rangle}$ | Very Very Big |
| $LAM_{\langle 0.9,0.2,0.3\rangle}$ | Less Absolute Maximum |
| $VB_{\langle 0.8,0.1,0.2\rangle}$ | Very Big |
| $Big_{\langle 0.7,0.2,0.3\rangle}$ | |
| $VM_{\langle 0.6,0.4,0.4\rangle}$ | Very Medium |
| $LV_{\langle 0.5,0.4,0.6\rangle}$ | Less Big |
| $VLB_{\langle 0.5,0,0\rangle}$ | Very Less Big |
| $VLS_{\langle 0.5,0.3,0.3\rangle}$ | Very Less Small |
| $M_{\langle 0.5,0.5,0.5\rangle}$ | MEDIUM |
| $LM_{\langle 0.4,0.7,0.8\rangle}$ | Less Medium |
| $LS_{\langle 0.4,0.4,0.4\rangle}$ | Less Small |
| $Small_{\langle 0.3,0.6,0.7\rangle}$ | |
| $VS_{\langle 0.2,0.7,0.8\rangle}$ | Very Small |
| $LAMin_{\langle 0.1,0.8,0.7\rangle}$ | Less Absolute Minimum |
| $VVS_{\langle 0.1,0.8,0.9\rangle}$ | Very Very Small |
| $0_{\langle 0,1,1\rangle}$ | ABSOLUTE MINIMUM |

## *XII.4.9. Conclusions*

In this paper, the classical hedge algebras have been extended for the first time to neutrosophic hedge algebras. With respect to an attribute, we have inserted the neutrosophic degrees of membership / indeterminacy / nonmembership of each generator, hedge, and constant. More than in the classical hedge algebras, we have introduced several numerical hedge operators: for hedge applied to element, and for hedge combined with hedge. An extensive example of a neutrosophic hedge algebra is given, and important properties related to it are presented.

## *References*

1. Lakoff, G. *Hedges, a study in meaning criteria and the logic of fuzzy concepts*. 8th Regional Meeting of the Chicago Linguistic Society, 1972.

2. Zadeh, L.A. *A fuzzy-set theoretic interpretation of linguistic hedges*. Journal of Cybernetics 1972, Volume 2, 04-34.

3. Cat Ho, N.; Wechler, W. *Hedge Algebras: An algebraic Approach to Structure of Sets of Linguistic Truth Values*. Fuzzy Sets and Systems 1990, 281-293.

## CHAPTER XIII: APPLICATIONS

### *XIII.1. Neutrosophic MCDM*

In neutrosophic multi-criteria decision making, instead of having crisp (positive number) values for the weights of the criteria, we have triplets (*t, i, f*)-values for the weights, where *t* is the degree of positive (in the qualitative sense, not in a numerical sense) value of a criterion weight, *i* is the degree of indeterminate value, and *f* is the degree of negative (in the qualitative sense) value of a criterion weight.

Of course, *t, i, f* are numbers (and in general subsets) of the interval [0, 1].

Similar for the neutrosophic alternatives, whose values are not crisp, but (*t, i, f*)-values.

## *XIII.2 Neutrosophic Psychology*

*Neutrosophic Psychology* means indeterminacy studied in psychology, and connection of opposite theories and their neutral theories together.

If a scale weights are, for example, 1, 2, 3, 4, 5, 6, 7, we can refine in many way, for example:

- pessimistically as T, $I_1$, $I_2$, $I_3$, $I_4$, $I_5$, F;
- optimistically as $T_1$, $T_2$, $I_1$, $I_2$, $I_3$, $F_1$, $F_2$;
- more optimistically $T_1$, $T_2$, $T_3$, I, $F_1$, $F_2$, $F_3$;

etc.

Surely, many ideas can be developed on the *refined* neutrosophic set.

## *XIII.3. Neutrosophic Function as The Equatorial Virtual Line*

There is an application of neutrosophic mathematical analysis (neutrosophic calculus) of which I would not have known without visiting Ecuador.

Equatorial imaginary line is actually a curve that circles the globe in the middle, called circumference, but it is not fixed, so it has a degree of indeterminacy, this curve ranging within a band (surface) with the width of 5 km surrounding the globe in the middle. Therefore, the equatorial line is a neutrosophic curve and analogously the Earth's circumference is a neutrosophic circumference. On a stretch of 5 km, it constantly varies due to changes in the physical forces of rotation, translation and mutation (periodic oscillation, inclination) of the Earth. As in neutrosophic logic, where the precise ... can be partially imprecise!

# *XIII.4. PCR5 and Neutrosophic Probability in Target Identification*[*]

## *Abstract*

In this paper, we use PCR5 in order to fusion the information of two sources providing subjective probabilities of an event *A* to occur in the following form: chance that *A* occurs, indeterminate chance of occurrence of *A*, chance that *A* does not occur.

## *XIII.4.1. Introduction*

Neutrosophic Probability [1] was defined in 1995 and published in 1998, together with neutrosophic set, neutrosophic logic, and neutrosophic probability.

The words "neutrosophy" and "neutrosophic" were introduced also in [1]. Etymologically, "neutrosophy" (noun) [French *neutre* < Latin *neuter*, *neutral*, and Greek *sophia*, skill/wisdom]

[*] In collab. with Nassim Abbas, Youcef Chibani, Bilal Hadjadji, Zayen Azzouz Omar from the University of Science and Technology, Algiers, Algeria.

means *knowledge of neutral thought*. While "neutrosophic" (adjective), means *having the nature of, or having the characteristic of Neutrosophy*.

Neutrosophy is a new branch of philosophy which studies the origin, nature, and scope of neutralities, as well as their interactions with different ideational spectra.

Zadeh introduced the degree of membership / truth ($t$) in 1965 and defined the fuzzy set.

Atanassov introduced the degree of nonmembership / falsehood ($f$) in 1986 and defined the intuitionistic fuzzy set.

Smarandache introduced the degree of indeterminacy/neutrality ($i$) as independent component in 1995 (published in 1998) and defined the neutrosophic set. In 2013, he refined / split the neutrosophic set to $n$ components: $t_1$, $t_2$, ...$t_j$; $i_1$, $i_2$, ..., $i_k$; $f_1$, $f_2$, ..., $f_l$, with $j+k+l = n > 3$. And, as particular cases of refined neutrosophic set, he split the fuzzy set truth into $t_1$, $t_2$, ...; and the intuitionistic fuzzy set into $t_1$, $t_2$, ... and $f_1$, $f_2$, ... .

See: http://fs.gallup.unm.edu/neutrosophy.htm.

For single valued neutrosophic logic, the sum of the components is:

- $0 \leq t+i+f \leq 3$ when all three components are independent;
- $0 \leq t+i+f \leq 2$ when two components are dependent, while the third one is independent from them;
- $0 \leq t+i+f \leq 1$ when all three components are dependent.

When three or two of the components T, I, F are independent, one leaves room for incomplete information (sum < 1), paraconsistent and contradictory information (sum > 1), or complete information (sum = 1).

If all three components T, I, F are dependent, then similarly one leaves room for incomplete information (sum < 1), or complete information (sum = 1).

## *XIII.4.2. Definition of Neutrosophic Measure*

A neutrosophic space is a set which has some indeterminacy with respect to its elements.

Let $X$ be a neutrosophic space, and $\Sigma$ a $\sigma$–neutrosophic algebra over $X$. A *neutrosophic measure* $\nu$ is defined by for neutrosophic set $A \in \Sigma$ by

$$\nu : X \to R^3,$$

$$\nu(A) = \big(m(A), m(neutA), m(antiA)\big), \qquad (1)$$

with *antiA* = the opposite of A, and *neutA* = the neutral (indeterminacy), neither A nor antiA (as defined above); for any $A \subseteq X$ and $A \in \Sigma$,

- m(A) means measure of the determinate part of A;
- m(neutA) means measure of indeterminate part of A;
- and m(antiA) means measure of the determinate part of antiA;

where $\nu$ is a function that satisfies the following two properties:

Null empty set: $\nu(\Phi) = (0,0,0)$.

Countable additivity (or σ-additivity): for all countable collections $\{A_n\}_{n \in L}$ of disjoint neutrosophic sets in $\Sigma$, one has:

$$\nu\left(\bigcup_{n\in L} A_n\right)=\left(\sum_{n\in L} m(A_n),\ \sum_{n\in L} m(neutA_n),\sum_{n\in L} m(antiA_n)-(n-1)m(X)\right) \quad (2)$$

where $X$ is the whole neutrosophic space, and

$$\sum_{n\in L} m(antiA_n)-(n-1)m(X)=m(X)-\sum_{n\in L} m(A_n)=m(\bigcap_{n\in L} antiA_n). \quad (3)$$

A neutrosophic measure space is a triplet $(X,\Sigma,\nu)$.

## *XIII.4.3. Normalized Neutrosophic Measure*

A neutrosophic measure is called normalized if

$$\nu(X)=(m(X),m(neutX),m(antiX))=(x_1,x_2,x_3), \quad (4)$$

with $x_1+x_2+x_3=1$, and $x_1\geq 0, x_2\geq 0, x_3\geq 0$, where, of course, $X$ is the whole neutrosophic measure space.

As a particular case of neutrosophic measure $\nu$ is the neutrosophic probability measure, i.e. a neutrosophic measure that measures probable/possible propositions

$$0\leq \nu(X)\leq 3,$$

where $X$ is the whole neutrosophic probability sample space.

For single valued neutrosophic logic, the sum of the components is:

- $0 \leq x_1+x_2+x_3 \leq 3$ when all three components are independent;
- $0 \leq x_1+x_2+x_3 \leq 2$ when two components are dependent, while the third one is independent from them;
- $0 \leq x_1+x_2+x_3 \leq 1$ when all three components are dependent.

When three or two of the components $x_1$, $x_2$, $x_3$ are independent, one leaves room for incomplete information (sum < 1), paraconsistent and contradictory information (sum > 1), or complete information (sum = 1).

If all three components $x_1$, $x_2$, $x_3$ are dependent, then similarly one leaves room for incomplete information (sum < 1), or complete information (sum = 1).

## *XIII.4.4. Normalized Probability*

We consider the case when the sum of the components m(A) + m(neutA) + m(antiA) =1.

We may denote the normalized neutrosophic probability of an event A as $NP(\mathcal{A}) = (t, i, f)$, where

t is the chance that $\mathcal{A}$ occurs, i is indeterminate chance of occurrence of $\mathcal{A}$, and f is the chance that $\mathcal{A}$ does not occur.

## *XIII.4.5. The PCR5 Formula*

Let the frame of discernment $\Theta = \{\theta_1, \theta_2, ..., \theta_n\}, n \geq 2.$ Let $G = (\Theta, \cup, \cap, C)$ be the super-power set, which is Θ closed under union, intersection, and respectively complement.

Let's consider two masses provided by 2 sources:

$m_1, m_2 : G \rightarrow [0, 1]$.

The conjunctive rule is defined as

$$m_{12}(X) = \sum_{X_1, X_2 \in G} m_1(X_1) m_2(X_2) \cdot \quad (5)$$

Then the Proportional Conflict Redistribution Rule (PCR) #5 formula for 2 sources of information is defined as follows:

$$\forall X \in G \setminus \{\phi\},$$

$$m_{PCR5}(X) = m_{12}(X) + \sum_{Y \in G \setminus \{X\}} \left[\frac{m_1(X)^2 m_2(Y)}{m_1(X) + m_2(Y)} + \frac{m_2(X)^2 m_1(Y)}{m_1(X) + m_2(Y)}\right] \quad (6)$$

where all denominators are different from zero.

If a denominator is zero, that fraction is discarded.

## *XIII.4.6. Application in Information Fusion*

Suppose an airplane $A$ is detected by the radar. What is the chance that $A$ is friend, neutral, or enemy?

Let's have two sources that provide the following information:

$NP_1^{(A)}(t_1, i_1, f_1)$, and $NP_2^{(A)}(t_2, i_2, f_2)$.

Then:

$$[NP_1 \oplus NP_2](t) = t_1 t_2 + \left(\frac{t_1^2 i_2}{t_1 + i_2} + \frac{t_2^2 i_1}{t_2 + i_1}\right) + \left(\frac{t_1^2 f_2}{t_1 + f_2} + \frac{t_2^2 f_1}{t_2 + f_1}\right) \quad (7)$$

Because: $t_1 i_2$ is redistributed back to the truth (t) and indeterminacy proportionally with respect to $t_1$ and respectively $i_2$:

$$\frac{x_1}{t_1} = \frac{y_1}{i_2} = \frac{t_1 i_2}{t_1 + i_2}, \quad (8)$$

whence

$$x_1 = \frac{t_1^2 i_2}{t_1 + i_2}, \; y_1 = \frac{t_1 i_2^2}{t_1 + i_2}. \quad (9)$$

Similarly, $t_2 i_1$ is redistributed back to $t$ and $i$ proportionally with respect to $t_2$ and respectively $i_1$:

$$\frac{x_2}{t_2} = \frac{y_2}{i_1} = \frac{t_2 i_1}{t_2 + i_1}, \quad (10)$$

whence

$$x_2 = \frac{t_2^2 i_1}{t_2+i_1}\ ,\ y_2 = \frac{t_2 i_1^2}{t_2+i_1}\ . \tag{11}$$

Similarly, $t_1 f_2$ is redistributed back to $t$ and $f$ (falsehood) proportionally with respect to $t_1$ and respectively $f_2$:

$$\frac{x_3}{t_1} = \frac{z_1}{f_2} = \frac{t_1 f_2}{t_1+f_2}\ , \tag{12}$$

whence

$$x_3 = \frac{t_1^2 f_2}{t_1+f_2}\ ,\ z_1 = \frac{t_1 f_2^2}{t_1+f_2}\ . \tag{13}$$

Again, similarly $t_2 f_1$ is redistributed back to $t$ and $f$ proportionally with respect to $t_2$ and respectively $f_1$:

$$\frac{x_4}{t_2} = \frac{z_2}{f_1} = \frac{t_2 f_1}{t_2+f_1}\ , \tag{14}$$

whence

$$x_4 = \frac{t_2^2 f_1}{t_2+f_1}\ ,\ z_2 = \frac{t_2 f_1^2}{t_2+f_1}\ . \tag{15}$$

In the same way, $i_1 f_2$ is redistributed back to $i$ and $f$ proportionally with respect to $i_1$ and respectively $f_2$:

$$\frac{y_3}{i_1} = \frac{z_3}{f_2} = \frac{i_1 f_2}{i_1+f_2}\ , \tag{16}$$

whence

$$y_3 = \frac{i_1^2 f_2}{i_1+f_2}\ ,\ z_3 = \frac{i_1 f_2^2}{i_1+f_2}\ . \tag{17}$$

While $i_2 f_1$ is redistributed back to $i$ and $t$ proportionally with respect to $i_2$ and respectively $f_1$:

$$\frac{y_4}{i_2} = \frac{z_4}{f_1} = \frac{i_2 f_1}{i_2 + f_1}, \tag{18}$$

whence

$$y_4 = \frac{i_2^2 f_1}{i_2 + f_1}, \; z_4 = \frac{i_2 f_1^2}{i_2 + f_1}. \tag{19}$$

Then

$$[NP_1 \oplus NP_2](i)$$

$$= i_1 i_2 + \left(\frac{i_1^2 t_2}{i_{11} + t_2} + \frac{i_2^2 t_1}{i_2 + t_1}\right) + \left(\frac{i_1^2 f_2}{i_1 + f_2} + \frac{i_2^2 f_1}{i_2 + f_1}\right), \tag{20}$$

and

$$[NP_1 \oplus NP_2](f)$$

$$= f_1 f_2 + \left(\frac{f_1^2 t_2}{f_1 + t_2} + \frac{f_2^2 t_1}{f_2 + t_1}\right) + \left(\frac{f_1^2 i_2}{f_1 + i_2} + \frac{f_2^2 i_1}{f_2 + i_1}\right). \tag{21}$$

## *XIII.4.7. Example*

Let's compute:

$(0.6, 0.1, 0.3) \wedge_N (0.2, 0.3, 0.5)$.

$t_1 = 0.6, i_1 = 0.1, f_1 = 0.3,$ and

$t_2 = 0.2, i_2 = 0.3, f_2 = 0.5,$

are replaced into the three previous neutrosophic logic formulas:

*(using PCR5 rule)*

$$[NP_1 \oplus nm_2](t) = 0.6(0.2) + \left(\frac{0.6^2(0.3)}{0.6+0.3} + \frac{0.2^2(0.1)}{0.2+0.1}\right) + \left(\frac{0.6^2(0.5)}{0.6+0.5} + \frac{0.2^2(0.3)}{0.2+0.3}\right) \simeq 0.44097.$$

$$[NP_1 \oplus NP_2](i) = 0.1(0.3) + \left(\frac{0.1^2(0.2)}{0.1+0.2} + \frac{0.3^2(0.6)}{0.3+0.6}\right) + \left(\frac{0.1^2(0.5)}{0.1+0.5} + \frac{0.3^2(0.3)}{0.3+0.3}\right) \simeq 0.15000.$$

$$[NP_1 \oplus NP_2](f) = 0.3(0.5) + \left(\frac{0.3^2(0.2)}{0.3+0.2} + \frac{0.5^2(0.6)}{0.5+0.6}\right) + \left(\frac{0.3^2(0.3)}{0.3+0.3} + \frac{0.5^2(0.1)}{0.5+0.1}\right) \simeq 0.40903.$$

*(using Dempster's Rule)*

Conjunctive Rule:

| | | |
|---|---|---|
| 0.12 | 0.03 | 0.15 |

Dempster's Rule:

| | | |
|---|---|---|
| 0.40 | 0.10 | 0.50 |

This is actually a PCR5 formula for a frame of discernment $\Omega = \{\theta_1, \theta_2, \theta_3\}$ whose all intersections are empty.

We can design a PCR6 formula too for the same frame.

Another method will be to use the neutrosophic $N-norm$, which is a generalization of fuzzy $T-norm$.

If we have two neutrosophic probabilities

| | **Friend** | **Neutral** | **Enemy** |
|---|---|---|---|
| $NP_1$ | $t_1$ | $i_1$ | $f_1$ |
| $NP_2$ | $t_2$ | $i_2$ | $f_2$ |

then

$$NP_1 \oplus NP_2 = (t_1 + i_1 + f_1) \cdot (t_2 + i_2 + f_2) =$$
$$t_1t_2 + t_1i_2 + t_2i_1 + i_1i_2 + t_1f_1 +$$
$$t_1f_2 + t_2f_1 + i_1f_2 + i_2f_1 + f_1f_2$$

Of course, the quantity of $t_1t_2$ will go to Friend, the quantity of $i_1i_2$ will go to Neutral, and the quantity of $f_1f_2$ will go to Enemy.

The other quantities will go depending on the pessimistic or optimistic way:

In the pessimistic way (lower bound) $t_1i_2 + t_2i_1$ will go to Neutral, and $t_1f_2 + t_2f_1 + i_1f_2 + i_2f_1$ to Enemy.

In the optimistic way (upper bound) $t_1i_2 + t_2i_1$ will go to Friend, and $t_1f_2 + t_2f_1 + i_1f_2 + i_2f_1$ to Neutral.

About $t_1f_2 + t_2f_1$, we can split it half-half to Friend and respectively Enemy.

We afterwards put together the pessimistic and optimistic ways as an interval neutrosophic probability.

Of course, the reader or expert can use different transfers of intermediate mixed quantities $t_1 i_2 + t_2 i_1$, and respectively $t_1 f_2 + t_2 f_1 + i_1 f_2 + i_2 f_1$ to Friend, Neutral, and Enemy.

## *XIII.4.8. Conclusion*

We have introduced the application of neutrosophic probability into information fusion, using the combination of information provided by two sources using the PCR5.

Other approaches can be done, for example the combination of the information could be done using the N-norm and N-conorm, which are generalizations of the T-norm and T-conorm from the fuzzy theory to the neutrosophic theory.

More research is needed to be done in this direction.

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# *XIII.5. Easier to Break a Dynamic System from Inside than from Outside*[*]

Almost all closed or open dynamic systems from our real world are closed or open neutrosophic dynamic systems, since they have indeterminacies – except the abstract or idealistic dynamic systems created as imaginary in pure theories.

A dynamic system, in general, is formed by a space, that comprises many elements and in between the elements there are some relationships.

There may be binary relationships (the most studied particular case), meaning relationships between only two elements, or in general $n$-ary relationships, for $n \geq 1$, which are called *hyper-relationships*, comprising all of them: relationships between an element and itself (for $n = 1$), binary relationships (for $n = 2$), ternary relationships (for $n = 3$), and so on.

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[*] In collab. with Andrusa R. Vatuiu.

If the dynamic system is open, then there also are (hyper)relationships between some inside elements with some outside elements. Almost all dynamic systems are open in some degree, since only theoretical dynamic systems can be considered as completely isolated from their environments.

The hyperrelationships are relationships of group, meaning that all elements into the group act together as a whole body.

If at least one of the space, elements, or hyperrelationships have some indeterminacy, we deal with a neutrosophic dynamic system.

Since the system is linearly or non-linearly dynamic, there are permanently changes with respect to the space (which may get bigger or smaller or may change its shape and position), with respect to its elements (which may partially belong, partially not belonging, and partially their belongness being indeterminate – and these belong-ness / non-belong-ness / indeterminacy may vary in time such that some elements may completely leave the system, while new elements

may enter into the system), and similarly the degrees of (hyper)relationships between interior elements among themselves, and the degrees of (hyper)relationships between interior and exterior elements may change too.

Let $\mathcal{U}$ be a universe of discourse. Let $\Omega$ be a space, $\Omega \subset \mathcal{U}$, that comprises the elements:

$\{x_1(T_1, I_1, F_1), x_2(T_2, I_2, F_2), \ldots, x_n(T_n, I_n, F_n)\}$,

for $n \geq 1$, and $T_i, I_i, F_i \subseteq [0,1]$, for $i \in \{1, 2, \ldots, n\}$,

where:

$T_i$ represents the degree of membership of the element $x_i$ with respect to the space $\Omega$;

$I_i$ represents the degree of indeterminate-appurtenance of the element $x_i$ with respect to the space $\Omega$; and

$F_i$ represents the degree of nonmembership of the element $x_i$ with respect to the space $\Omega$.

Hence $\Omega$ is a *neutrosophic space* (set).

Let a *neutrosophic open/closed hyperrelationship* be defined as:

$$\mathcal{R}_{HR}: \Omega^k \times \mathcal{C}(\Omega)^l \to \mathcal{P}([0,1])^3 \qquad (1)$$

$$\mathcal{R}_{HR}\left(x_{i_1}, x_{i_2}, \ldots, x_{i_k}, y_{j_1}, y_{j_2}, \ldots, y_{j_l}\right) = (T_{\mathcal{R}}, I_{\mathcal{R}}, F_{\mathcal{R}}),$$

which means that the open hyperrelationship between the inside elements $x_{i_1}, x_{i_2}, \dots, x_{i_k} \in \Omega$, and outside elements $y_{j_1}, y_{j_2}, \dots, y_{j_l} \in \mathcal{C}(\Omega)$, where $\mathcal{C}(\Omega)$ is the neutrosophic complement of $\Omega$ with respect to the universe of discourse $\mathcal{U}$, has the neutrosophic truth-value $(T_{\mathcal{R}}, I_{\mathcal{R}}, F_{\mathcal{R}})$, where $T_{\mathcal{R}}, I_{\mathcal{R}}, F_{\mathcal{R}} \subseteq [0,1]$; and $k$ may vary between 1 and $n$, also $l$ may vary between 0 and $card(\mathcal{C}(\Omega))$, i.e. cardinal (number of elements) of $\mathcal{C}(\Omega)$. When $l = 0$ we have only interior (inside) hyperrelationship, and the system is considered closed. If $l \geq 1$, we have exterior (outside) hyper-relationship, and the system is considered open.

Therefore:

$$D_N = \left(\Omega, \{x_i(T_i, I_i, F_i), i \in \{1, 2, \dots, n\}\}, \mathcal{R}_{HR}, HR \subset L\right)$$

where L is the set of all possible neutrosophic open/closed hyperrelationships on $\Omega$, is a *neutrosophic complex dynamic system*.

## *XIII.5.1. Modeling Methodology.*

A real world open dynamic system is abstracted to a mathematical model. The unity and dis-unity of the open dynamic system changes over time,

and this influences the stability and the instability of the system.

This is an analytical model that tries to approximately replicate the mechanism of the open dynamic system, using ODEs (ordinary differential equations).

We make the following assumptions:

- All the initial values (parameters) are positive constants.
- The interactions (hyperrelationships) among inside elements of the system, or among inside and outside elements occur in a homogeneous way.
- The inside elements have neutrosophic degrees $(T, I, F)$ of appurtenance to the system (population).
- Similarly, the outside elements have neutrosophic degrees of appurtenance to the complement of the system (the outside world).
- At the start (when time $t = 0$), the open dynamic system is considered in equilibrium (or stable).

– The system is not directly attacked from outside.

## *XIII.5.2. Model of Breaking a Neutrosophic Complex Dynamic System*

Similar to modeling the Biological Immune Dynamic System in response to the pathogen organisms, or to the Prey-Predator Dynamic System, or to the Computer Network Dynamic System in response to the propagation of worms, viruses, Trojans and Backdoors, we propose a model to simulate the breaking up of neutrosophic complex dynamic system using Ordinary Differential Equations (ODE).

*Agent-Based Models* and *Cellular Automata* can also be proposed to simulate the breaking up of a (neutrosophic) complex dynamic system.

We use *variables* to describe, as functions of time ($t$), specific attributes of the population (totality of elements) of the space $\Omega$.

We also use *parameters* to describe initial quantities, rates, and constants with respect to the population.

1) Let $A$ be the total initial number of inside individuals (elements) $x_i\left(T_i^{\Omega}, I_i^{\Omega}, F_i^{\Omega}\right) \in \Omega$ such that $\sup T_i > 0$, meaning that $x_i$ has some non-zero positive degree of membership with respect to $\Omega$, where $\left(T_i^{\Omega}, I_i^{\Omega}, F_i^{\Omega}\right)$ is the neutrosophic truth-value of $x_i$ with respect to $\Omega$.

Let $\alpha(t)$ be the variable that describes the population at time $t$. Let $a_1$be the constant rate at which new individuals not in hyperrelationships with outsiders are partially or totally added to the system. And let $a_2$ be the constant rate at which individuals not in hyperrelationships with outsiders leave the system.

«Partially or totally» means that the neutrosophic membership degree $(T, I, F)$, with respect to the system, has $\sup T > 0$. «Leaving the system» means that the neutrosophic membership degree $(T, I, F)$, with respect to the system, has $\sup T = 0$.

2) Let $B$ be the total initial number of outside individuals $y_j\left(T_j^{\mathcal{C}}, I_j^{\mathcal{C}}, F_j^{\mathcal{C}}\right) \in \mathcal{C}(\Omega)$, with $\sup T_j > 0$, where $\left(T_j^{\mathcal{C}}, I_j^{\mathcal{C}}, F_j^{\mathcal{C}}\right)$ is the neutrosophic truth-value

of $y_j$ with respect to $\mathcal{C}(\Omega)$, such that $\mathcal{R}_H(\dots x_i \dots y_j \dots) = (T_{\mathcal{R}}, I_{\mathcal{R}}, F_{\mathcal{R}})$, with $supT_{\mathcal{R}} > 0$. These are outside individuals that have some neutrosophic hyperrelationships with inside individuals.

Let $\beta(t)$ be the variable that describes the number of outside individuals that have some neutrosophic hyperrelationships with inside individuals.

Let $b_1$ be the constant rate at which new outside individuals partially or totally get into neutrosophic hyperrelationships with insiders, while $b_2$ be the constant rate at which new old outsiders leave the neutrosophic hyperrelationships with insiders.

Let $b_3$ be the constant rate at which new inside individuals partially or totally get into neutrosophic hyperrelationships with outsiders, while $b_4$ be the constant rate at which new old insiders leave the neutrosophic hyperrelationships with outsiders.

3) Let $C$ be the total initial number of outside individuals not involved in open hyperrelationships. An individual (element) $y_j(T_j^{\mathcal{C}}, I_j^{\mathcal{C}}, F_j^{\mathcal{C}})$

is considered outside of the $D_N$ if its membership $T_j^{\mathcal{C}}$ with respect to $\mathcal{C}(\Omega)$ has $\sup T_j^{\mathcal{C}} > 0$ ,while its membership $T_j^{\Omega}$, with respect to $\Omega$, has $\sup T_j^{\Omega} = 0$, where $y_j\left(T_j^{\Omega}, I_j^{\Omega}, F_j^{\Omega}\right)$ is its neutrosophic truth-degree with respect to $\Omega$.

Let $\gamma(t)$ be the variable that describes the number of outside individuals, not involved in open hyper-relations with inside individuals. Let $c_1$ be the constant rate at which new outside individuals, not involved in open hyperrelationships with inside individuals, are added to $\mathcal{C}(\Omega)$; while $c_2$ be the constant rate at which old outside individuals, not involved in open hyperrelationships with inside individuals, leave the $\mathcal{C}(\Omega)$.

4) Let $D$ be the initial number of the inside individuals of the system, not involved in open hyperrelationships, that act as sneaks / spies / boycotters for the enemy.

Let $\delta(t)$ be the variable describing the number of inside individuals not involved in open hyperrelationships turned to sneaks / spies / boycotters for the enemy.

Let $d_1$ be the constant rate at which new insiders not involved in open relationships are recruiting as sneaks / spies / boycotters for the enemy.

Let $d_2$ be the constant rate at which old sneaks / spies / boycotters not involved in hyper-relationships cease to be sneaks / spies / boycotters for the enemy.

5) Let $E$ be the total initial number of outside enemy intruders, e.g. hostile individuals, corporations, societies, companies, publications, mass-media(tors), ideology, enemy politics, linguistics, invasive culture / traditions, influence agents, etc., acting as spies, boycotters, denigrators (not involved in hyperrelationships), acting partially or totally against the system.

Let $\eta(t)$ be the variable describing the number of enemy intruders (not involved in open hyper-relationships) at time $t$ acting as spies or boycotters.

Let $e_1$ be the constant rate at which new enemy intruders are partially or totally added to the system, acting as spies or boycotters.

Let $e_2$ be the constant rate at which new enemy intruders (not involved in open hyperrelationships) cease to be sneaks, spies, boycotters of the system.

*Universe of Discourse*

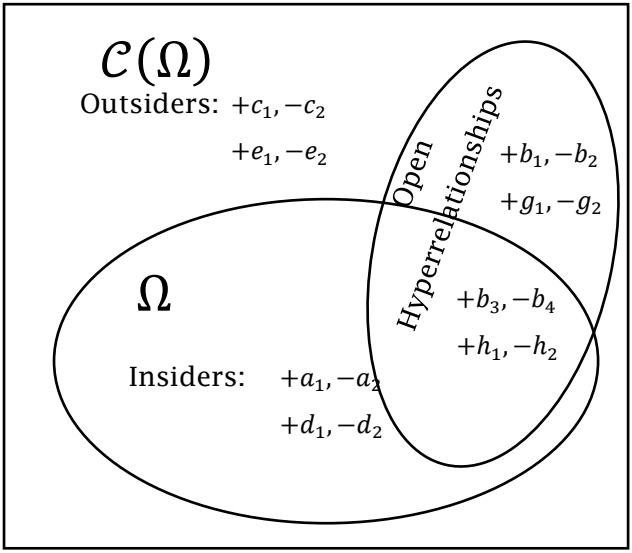


*Diagram 1 of Breaking a Neutrosophic Open Complex System*

6) Let $G$ be total initial number of outside enemy intruders involved in open hyper-

relationships acting as sneaks / spies / boycotters against the system. Let $\mu(t)$ be the variable describing the number of outside individuals, involved in open hyperrelationships, that act as sneaks / spies / boycotters against the system.

Let $g_1$ be the constant rate at which new outside enemy intruders involved in open hyper-relationships acting as sneaks / spies / boycotters are added, and $g_2$ be the constant rate at which old outside enemy intruders involved in open hyperrelationships cease to be sneaks / spies / boycotters against the system.

7) Let $H$ be total initial number of inside individuals, involved in open hyperrelationships, acting as sneaks / spies / boycotters against the system.

Let $\nu(t)$ be the variable describing the number of inside individuals, involved in open hyper-relationships, that act as sneaks / spies / boycotters against the system.

Let $h_1$ the constant rate at which new inside individuals, involved in open hyperrelationships, act as sneaks / spies / boycotters against the

system, and $h_2$ be the constant rate at which old inside individuals, involved in open hypper-relationships, cease to act as sneaks / spies / boycotters against the system.

8) *Neutrosophic Probabilities defined on the Neutrosophic Open Complex Dynamic System.*

In order to better describe the behavior of a neutrosophic open complex dynamic system, let's provide the following definition:

The neutrosophic probability of an event $E$ in general is $P(E) = \big(Ch(E)\big), Ind(E), NonCh(E))$, with:

$Ch(E)$ = chance that event $E$ occurs;

$Ind(E)$ = indeterminate-chance that event $E$ occurs = $Ch(neutE)$;

$NonCh(E)$ = chance that event $E$ does not occur = $Ch(antiE)$.

One may also write:

$P(E) = \big(Ch(E)\big), Ch(neutE), Ch(antiE))$

with $Ch(E), Ch(neutE), Ch(antiE) \subseteq [0,1]$.

In this paper, we consider the particular case when $Ch(E)$, $Ch(neutE)$, and $Ch(antiE) \in [0,1]$, i.e. we use the single-valued neutrosophic probability.

Let $p_1$ be the neutrosophic probability of recruiting sneaks / spies / boycotters from the new $a_1$ inside individuals, not involved in open hyperrelationships. And $p_2$ the neutrosophic probability that among the old $a_2$ inside individuals not involved in open hyperrelationships that left were sneaks / spies / boycotters.

Let $p_3$ be the neutrosophic probability of recruiting sneaks / spies / boycotters from the new $b_3$ inside individuals that are involved in open hyper-relationships. And $p_4$ the neutrosophic probability that from the old $b_4$ inside individuals, involved in open hyperrelationships, were sneaks / spies / boycotters.

Let $p_5$ be the neutrosophic probability of recruiting sneaks / spies / boycotters from the new $b_1$ outside individuals, involved in open hyperrelationships. And let $p_6$ be the neutrosophic probability that from the old outside individuals, involved in open hyper-relationships, there were sneaks / spies / boycotters.

Let $p_7$ be the neutrosophic probability of recruiting sneaks / spies / boycotters from the

new $c_1$ inside individuals, not involved in open hyperrelationships. And let $p_8$ be the neutrosophic probability that from the old $c_2$ outside individuals, not involved in open hyperrelationships, were sneaks / spies / boycotters.

9) *Spying/ Boycotting (Anti-System) Variables.*

The independent variable is time ($t$). All other variables are dependent on $t$. They are: $\alpha(t)$, $\beta(t)$, $\gamma(t)$, $\delta(t)$, $\eta(t)$, $\mu(t)$, $\nu(t)$, defined previously, and three more dependent variables defined below: $S_1(t), S_2(t)$ and $S(t)$.

Let $S_1(t)$ represent the variable describing the total number of inside sneaks / spies / boycotters:

$$S_1(t) = \delta(t) + \nu(t), \tag{2}$$

with the initial value

$$S_1(0) = \delta(0) + \nu(0) = D + H. \tag{3}$$

Let $S_2(t)$ be the variable describing the total number of outside spies / boycotters intruded into the system:

$$S_2(t) = \mu(t) + \eta(t), \tag{4}$$

with initial value

$$S_2(0) = \mu(0) + \eta(0) = G + E. \tag{5}$$

Let $S(t)$ represent the variable describing the total number of inside and outside intruders / spies / boycotters, together with their actions (hyper-relationships) against the system:

$$S(t) = S_1(t) + S_2(t) = \delta(t) + \nu(t) + \mu(t) + \eta(t), \quad (6)$$

with initial value

$$S(0) = S_1(t) + S_2(0) = D + H + G + E. \quad (7)$$

## *XIII.5.3. Ordinary Differential Equations Model*

We propose a system of ordinary differential equations.

$$\frac{dS_1}{dt} = \frac{d\delta}{dt} + \frac{d\nu}{dt} = [d_1 \cdot \delta(t) - d_2 \cdot \delta(t) + p_1 a_1 \cdot \alpha(t) - p_2 a_2 \cdot \alpha(t)] + [h_1 \cdot \nu(t) - h_2 \cdot \nu(t) + p_3 b_3 \beta(t) - p_4 b_4 \beta(t)] = (d_1 - d_2) \cdot \delta(t) + (h_1 - h_2) \cdot \nu(t) + (p_1 a_1 - p_2 a_2) \cdot \alpha(t) + (p_3 b_3 - p_4 b_4) \cdot \beta(t),$$

with $S_1(0) = D + H$. (8)

$$\frac{dS_2}{dt} = \frac{d\mu}{dt} + \frac{d\eta}{dt} = [g_1 \cdot \mu(t) - g_2 \cdot \mu(t) + p_5 \cdot b_1 \cdot \beta(t) - p_6 \cdot b_2 \cdot \beta(t)] + [e_1 \cdot \eta(t) - e_2 \cdot \eta(t) + p_7 \cdot c_1 \cdot \gamma(t) - p_8 \cdot c_2 \cdot \gamma(t)] = (g_1 - g_2) \cdot \mu(t) + (e_1 - e_2) \cdot \eta(t) + (p_5 b_1 - p_6 b_2) \cdot \beta(t) + (p_7 \cdot c_1 - p_8 \cdot c_2) \cdot \gamma(t),$$

with $S_2(0) = G + E$. (9)

Hence: $\frac{dS}{dt} = \frac{dS_1}{dt} + \frac{dS_2}{dt} = (d_1 - d_2) \cdot \delta(t) + (h_1 - h_2) \cdot \nu(t) + (g_1 - g_2) \cdot \mu(t) + (e_1 - e_2) \cdot \eta(t) + (p_1 a_1 - p_2 a_2) \cdot \alpha(t) + (p_3 b_3 - p_4 b_4 + p_5 b_1 - p_6 b_2) \cdot \beta(t) + (p_7 c_1 - p_8 c_2) \cdot \gamma(t)$,

with $S(0) = D + H + G + E$. (10)

## *XIII.5.4. Operations with Single-Valued Neutrosophic Probabilities*

Since $p_i$, for $1 \leq i \leq 8$, are vectors of the form

$$p_i = \big(Ch(E_i), Ch(neutE_i), Ch(antiE_i)\big), \quad (11)$$

where $E_i$ are events, and $Ch(E_i)$, $Ch(neutE_i)$, $Ch(antiE_i)$ are single-valued numbers in $[0,1]$, we use the following operations with such triads: for all $\psi, u_1, v_1, w_1, u_2, v_2, w_2 \in \mathbb{R}$, one has

$$(u_1, v_1, w_1) + (u_2, v_2, w_2) = (u_1 + u_2, v_1 + v_2, w_1 + w_2) \quad (12)$$

$$(u_1, v_1, w_1) - (u_2, v_2, w_2) = (u_1 - u_2, v_1 - v_2, w_1 - w_2) \quad (13)$$

$$\psi \cdot (u_1, v_1, w_1) = (\psi u_1, \psi v_1, \psi w_1) \quad (14)$$

$\psi + (u_1, v_1, w_1) = \psi \cdot (1,0,0) + (u_1, v_1, w_1) =$

$(\psi, 0, 0) + (u_1, v_1, w_1) = (\psi + u_1, v_1, w_1)$. (15)

## *XIII.5.5. Operations with Subset-Valued Neutrosophic Probabilities*

In the case when the above $u_1, v_1, w_1, u_2, v_2, w_2$ are subsets of $[0,1]$ one has:

$$(u_1, v_1, w_1) \oplus (u_2, v_2, w_2) = (u_1 \oplus u_2, v_1 \oplus v_2, w_1 \oplus w_2) \tag{16}$$

$$(u_1, v_1, w_1) \ominus (u_2, v_2, w_2) = (u_1 \ominus u_2, v_1 \ominus v_2, w_1 \ominus w_2) \tag{17}$$

$$\psi \odot (u_1, v_1, w_1) = (\psi \odot u_1, \psi \odot v_1, \psi \odot w_1) \tag{18}$$

where $\psi \in \mathbb{R}$

$$\psi \oplus (u_1, v_1, w_1) = \psi \cdot (1,0,0) \oplus (u_1, v_1, w_1) = (\psi, 0, 0) \oplus (u_1, v_1, w_1) = (\psi \oplus u_1, v_1, w_1) \tag{19}$$

And, of course:

$$u_1 \oplus u_2 = \{x + y | x \in u_1, y \in u_2\} \tag{20}$$

$$u_1 \ominus u_2 = \{x - y | x \in u_1, y \in u_2\} \tag{21}$$

$$\psi \odot u_1 = \{\psi \cdot x | x \in u_1\} \tag{22}$$

$$\psi \oplus u_1 = \{\psi + x | x \in u_1\} \tag{23}$$

which are: addition of subsets, subtraction of subsets, multiplication with a scalar of a subset, and addition of a scalar to a subset respectively. For $v_1, v_2, w_1, w_2$, the same operations.

Of course, we restrict all operations' results to the interval [0, 1]. If a result is < 0, we write 0 instead, and if the result is > 1, we write 1 instead.

## *XIII.5.6. Whole Neutrosophic Hyperrelationships*

Let $\mathcal{R}_{nonS}$ be the whole neutrosophic hyperrelationship of the $\Omega$ neutrosophic space (only inside individuals that are not sneaks, spies, boycotters for the enemy of the system), together with the outside individuals that are in open hyperrelationships with insiders, and such outsiders that are not sneaks, spies, boycotters against the system. "nonS" means "non-spies, non-boycotters etc.".

This hyperrelationship represents the cumulated power of all positive elements (individuals) of the population of $\Omega$, together with all positive (qualitatively) outside individuals, and all of their connections or hyperrelationships as the edges or hyperedges in the following neutrosophic hypergraph representing our neutrosophic complex dynamic system:

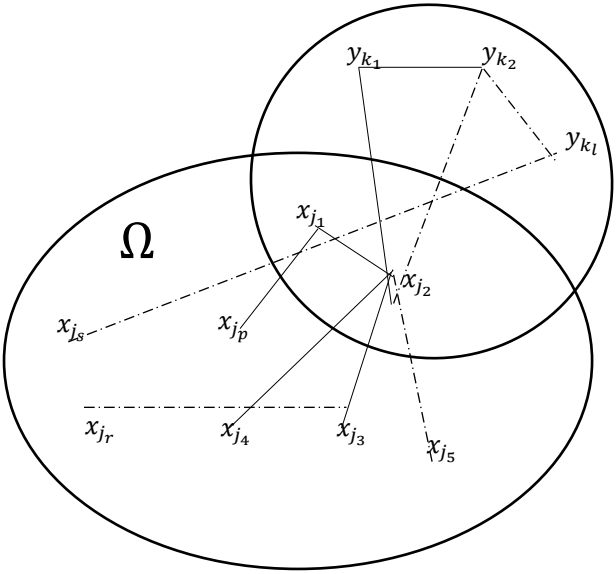


*Diagram 2 of* $\mathcal{R}_{nonS}$

where the hyperrelationship between nodes (individuals) is of neutrosophic form:

$$\mathcal{R}\left(x_{j_1}, x_{j_2}, \dots, x_{j_p}, x_{j_r}, x_{j_s}, y_{k_1}, y_{k_2}, \dots, y_{k_l}\right) =$$
$$= \left(t_{j_1 \dots j_s k_1 \dots k_l}, i_{j_1 \dots j_s k_1 \dots k_l}, f_{j_1 \dots j_s k_1 \dots k_l}\right)$$
$$\subseteq ([0,1],[0,1],[0,1]) \qquad (24)$$

for all $x_{j_1}, x_{j_2}, \dots, x_{j_p}, x_{j_r}, x_{j_s} \in \{x_1, x_2, \dots, x_n\} \subseteq \Omega$, and all $y_{k_1}, y_{k_2}, \dots, y_{k_l} \in \mathcal{C}(\Omega)$.

The $\mathcal{R}_{nonS}$ represents the maximum possible power (militarily, economically, financially,

adminis-tratively, politically, ideologically, etc.) of the neutrosophic dynamic system.

This occurs when it is a perfect unity among insiders themselves and perfect unity in the open hyperrelationships between insiders and outsiders.

Let's denote this maximum power by $m_{nonS}$.

Consequently, one has an obvious:

## *XIII.5.7. Theorem*

To destroy, or conquer, or break a neutrosophic dynamic system from outside, another neutrosophic dynamic system is needed whose maximum power is greater than $m_{nonS}$.

*

Unfortunately, in practice, such perfect unities are unrealistic in our world.

Let $\mathcal{R}_{D_N}$ be the whole neutrosophic hyper-relationship of the whole $\Omega$ neutrosophic space (all inside individuals, which are or which are not sneaks, spies, boycotters on behalf of the enemy), together with the outside individuals being in open hyper-reationships with inside individuals

(that are or that are not sneaks, spies, boycotters on behalf of the enemy).

The open hyperrelationship leave higher chances for outsiders and insiders for making system backdoors that help breaking the system from inside.

Obviously, the maximum possible power of $\mathcal{R}_{D_N}$, denoted by $m_{D_N}$, is strictly smaller than the previous one:

$$m_{D_N} < m_{nonS},$$

since the inside and outside spies work against the system, diminishing its power.

Unity means power, and split-ness means weakness. As in the well-knows Latin aphorism: *Divide et impera*.

## *XIII.5.8. Breaking Point Equilibrium Threshold*

The variable S(t) describes the total number of inside and outside individual that are sneaks, spies, boycotters, together with their actions (hyper-relationships) against the system, at time $t \geq 0$.

These individuals and their actions constitute the *negative qualitatively power against the system*. Let's denote it by $m_S$.

Therefore:

$$m_{D_N} = m_{nonS} - m_S. \quad (25)$$

For each neutrosophic dynamic system $D_N$ there is a *Breaking Point* or *Equilibrium Threshold*, $\tau_{D_N}$, where the system breaks down (collapses) if $m_S > \tau_{D_N}$ or the negative qualitatively power against the system overpasses the equilibrium threshold.

One has the following situations (when no direct attack from outside occurs):

If $m_S < \tau_{D_N}$ the system is in equilibrium (it is stable);

If $m_S = \tau_{D_N}$ the system is on the edge (between stability and instability);

If $m_S > \tau_{D_N}$ the system is breaking down from inside (it got instable).

An outside power $m_{out} > m_{D_N}$ is needed to be able to break the system from outside. $\tau_{D_N}$ depends on the type of dynamic system, its

structure and hyperrelationships (functionality), alike a construction scaffolding that may fell down when some key-links are broken...

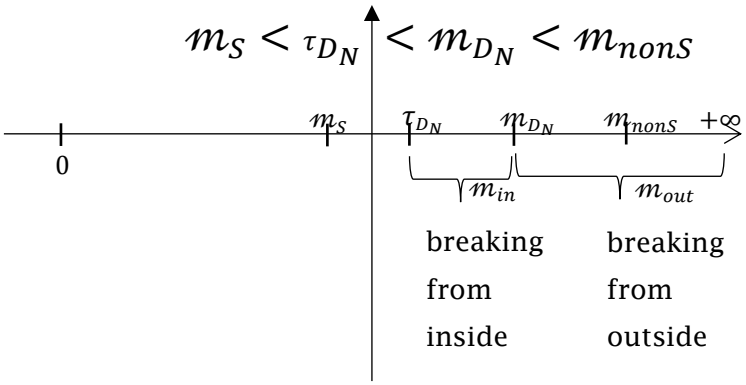


*Diagram 3 of A Dynamic System Breaking from Inside or from Outside.*

While only this inside power $m_{in} \in (\tau_{D_N} m_{D_N}]$ is needed to break the system from inside. Therefore:

$m_{in} \leq m_{D_N} < m_{out}$.

Therefore, it is easier to break a system from inside, than from outside. In order to do this, the inside force has to exceed a critical value (the

Equilibrium Threshold) to rich the system's dysfunctionality.

The smallest force needed to break down from outside a neutrosophic complex dynamic system is greater than the biggest force needed to break it down from inside.

In practice, the needed force from inside (by defectors, intruders, detractors, paid foreign agents, spies, instigators, and in general anti-system individuals) is much smaller than the needed force from outside used to destroy the system.

The percentage of anti-system inside population and the intensity of their anti-system actions count towards the breaking of the system from inside. In general, a system is broken by simultaneous attacks from both inside and outside the system.

The attack from inside helps lightening the attack from outside.

Breaking (or Attacking) from inside a neutrosophic complex linear or non-linear dynamic system, in general, is similar (in a

particular case), to a Cyber War: penetrating and destroying a computer network with *worms* (malicious codes which infect the computer system), *viruses* (which self-replicate), and mostly with *Trojan Horses* (which are programs that preform secretive operations (i.e. data being changed, stolen, deleted, or fake data included, or destructive executables added to the computer operation system), secretive operations under the mask of a legitim program), or creating *Backdoors* (where the inside and outside attacks can go through.

No neutrosophic dynamic system is 100% percent immune to intruders and boycotters, since such system has some indeterminacy, where there may be set up Backdoors.

We may see cyber-assaults, cyber-crimes, and global cyber-shocks from outside and from inside the system. If the anomaly into the system has very little impact, it is hard to detect. Abnormal and suspicious activities should be checked. The risk management is necessary in order to estimate

the digital threats, and to detect them as soon as possible.

A neutrosophic dynamic system has a degree of *vulnerability*, a degree of *invulnerability* (immunity), and degree of *indeterminacy* (unsurety if it's vulnerability or invulnerability). It functions under a certain risk tolerance level. Any neutrosophic dynamic system can be infiltrated. The more and more porous become the system's boundaries, the easier, faster, and more massive it can be infiltrated. Lone-wolf attacker is more difficult to detect.

## *XIII.5.9. Examples of Complex Dynamic System*

A complex dynamic system may be any association, organization, company, corporation, firm, farm, factory, team, country, empire, geographic area, digital or non-digital network, and so on.

## *XIII.5.10. Methods Used for Breaking from Inside a Complex Dynamic System*

– Interpreting what is good as bad, and praising what is bad;

- Reversing the value scale;
- Promoting within the system the non-values;
- Favoring the counter-selection for the all sectors of activities;
- Installing puppet leaders and puppet associates;
- Creating conspiracies and coups d'états;
- Using lone-wolf attackers that are harder to detect;
- Setting all individuals against each other within the system;
- Promotion for political reasons;
- Encouraging the incompetence and persecuting the competence;
- Encouraging self-disorganization;
- Making individuals hate themselves and their origin;
- Promoting the apathy of individuals with respect to extraneous intrusion;
- Using subservient media for anti-system propaganda;

- Boycotting everything positive within the system in economy, finance, administration;
- Making regulation that ignore or undermine and ridicule local tradition, culture, religion, education, health;
- Using disinformation and fake information;
- Transforming the system into a rigid (not flexible) one: not self-learning, nor self-adopting to environment;
- Increasing the system vulnerability and decrease its immunity;
- Obscuring the distinction between system normal behavior and misbehavior;
- Making the system unprepared for defense by depraving and annihilating its defense;
- Exaggerating the system's negations and diminishing or ignoring its positives;

- Biased predictions and fake statistics;
- Fraudulent elections;
- Any neutrosophic dynamic system has a degree of openness to outside, a degree to closeness; and a degree of indeterminate openness-closeness; the more open is the system to outside, the easier is to break it;
- The more the insiders are connected to the outsiders, the easier to break the system;
- The attackers should change all the times their breaking strategies;
- Using outside attack from within;
- Recompensing and rewarding null persons, system defectors, spies, sneaks, and the anti-system individuals;
- Imprisoning or denigrating pro-system individuals;
- Discouraging the order, promoting the anarchy;

- Making the system's boundaries between inside and outside vaguer and vaguer, so it can be better penetrated;
- Extending the system's insecurity zone;
- Creating hidden holes in the system's defense wall;
- Open gaps into the system;
- Spreading anti-system feelings, anti-socially engineered events, chaotic phenomena, dis-structure;
- To real problems bringing anti-solutions;
- Using the paradoxism into the system: what is $\langle A \rangle$, where $\langle A \rangle$ represents an entity (idea, notion, activity, attribute, etc.), should be interpreted as its opposite $\langle antiA \rangle$, and reciprocally;
- Even more general, use neutrosophism into the system: what is $\langle A \rangle$ interpret as $\langle neutA \rangle$ or $\langle antiA \rangle$, where

⟨*neutA*⟩ is the neutral: neither ⟨*A*⟩, nor ⟨*antiA*⟩;

- and reciprocally, what is ⟨*antiA*⟩ should be interpreted as ⟨*neutA*⟩ or ⟨*A*⟩;
- for example: ignore [i.e. ⟨*neutA*⟩] the worthy local personalities [*i.e.*⟨*A*⟩], or discredit [*i.e.*⟨*antiA*⟩] them.
- The paradoxism and neutrosophism are abstractizations and generalizations of Sun Tzu's ideas.

## *XIII.5.11. Extension of the Model*

The accuracy of the system can be increased if the mathematical constants, used into the model below, are extended to functions of time, i.e.:

$a_1 \to a_1(t)$, $a_2 \to a_2(t)$;

$b_1 \to b_1(t)$, $b_2 \to b_2(t)$, $b_3 \to b_3(t)$, $b_4 \to b_4(t)$;

$c_1 \to c_1(t)$, $c_2 \to c_2(t)$;

$d_1 \to d_1(t)$, $d_2 \to d_2(t)$;

$e_1 \to e_1(t)$, $e_2 \to e_2(t)$;

$g_1 \to g_1(t)$, $g_2 \to g_2(t)$;

$h_1 \to h_1(t)$, $h_2 \to h_2(t)$.

## *XIII.5.12. Equilibrium Points*

Are points where the derivatives of the variables are equal to zero, therefore, the variables do not change with respect to time:

$\frac{d\alpha}{dt} = 0, \frac{d\beta}{dt} = 0, \frac{d\gamma}{dt} = 0, \frac{d\delta}{dt} = 0, \frac{d\eta}{dt} = 0, \frac{d\mu}{dt} = 0,$

$\frac{d\nu}{dt} = 0, \frac{dS_1}{dt} = 0, \frac{dS_2}{dt} = 0,$ and $\frac{dS}{dt} = 0.$

## *XIII.5.13. Comments on the Model*

– If the entry constants are correspondingly equal to their exit constants (or $a_1 = a_2$, $b_1 = b_2$, $c_1 = c_2$, $d_1 = d_2$, $e_1 = e_2$, $g_1 = g_2$, and $h_1 = h_2$) and their corresponding neutrosophic probabilities of containing antisystem individuals (or $p_1 = p_2$, $p_3 = p_4$, $p_5 = p_6$, and $p_7 = p_8$) then $\frac{dS}{dt} = 0$ and the dynamic system is in equilibrium.

– If $\frac{dS}{dt} < \tau_{D_N}$, the system remains resistant to the attack from inside, and in equilibrium.

– If $\frac{dS}{dt} = \tau_{D_N}$, the system riches the breaking point.

– If $\frac{dS}{dt} > \tau_{D_N}$, the system is broken from inside, and gets in disequilibrium (instability).

– If $\lim_{t\to\infty}\left(\frac{dS}{dt}\right) = 0$, the system is in global asymptotical stability.

## *XIII.5.14. Conclusion*

This paper defines a neutrosophic mathematical model using a system of ordinary differential equations and the neutrosophic probability in order to approximate the process of breaking from inside a neutrosophic complex dynamic system. It shows that for breaking from inside it is needed a smaller force than for breaking from outside the neutrosophic complex dynamic system. Methods that have been used in the past for breaking from inside are listed. Simulation and animation of this neutrosophic dynamical system are needed for the future since, by changing certain parameters, various types of breaking from inside may be simulated.

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*Disclaimer.*

This paper does not advise anybody to break a dynamic system from inside, nor from outside.

This paper is only an attempt of making an approximate mathematical model of dynamic systems broken from inside in the past, and the paper lists several methods that have been used.

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# *XIII.6. Neutrosophic Quantum Computer*

*Abstract.*

This paper is a theoretical approach for a potential neutrosophic quantum computer to be built in the future, which is an extension of the classical theoretical quantum computer, into which the indeterminacy is inserted.

## *XIII.6.1. Introduction.*

Neutrosophic quantum communication is facilitated by the neutrosophic polarization, that favors the use the neutrosophic superposition and neutrosophic entanglement.

The neutrosophic superposition can be linear or non-linear. While into the classical presumptive quantum computers there are employed only the coherent superpositions of two states (*0* and *1*), in the neutrosophic quantum computers one supposes the possibilities of using *coherent superpositions amongst three states* (*0, 1,* and *I* =

indeterminacy) and one explores the possibility of using the *decoherent superpositions* as well.

## *XIII.6.2. Neutrosophic Polarization.*

The *neutrosophic polarization* of a photon is referred to as orientation of the oscillation of the photon: oscillation in one direction is interpreted as *0*, oscillation in opposite direction is interpreted as *1*, while the ambiguous or unknown or vague or fluctuating back and forth direction as *I* (indeterminate).

Thus, the neutrosophic polarization of a photon is *0, 1,* or *I*. Since indeterminacy (*I*) does exist independently from *0* and 1, we cannot use fuzzy nor intuitionistic fuzzy logic / set, but neutrosophic logic / set.

These three neutrosophic values are used for *neutrosophically encoding* the data.

## *XIII.6.3. Refined Neutrosophic Polarization.*

In a more detailed development, one may consider the *refined neutrosophic polarization*, where we refine for example *I* as $I_1$ (ambiguous

direction), $I_2$ (unknown direction), $I_3$ (fluctuating direction), etc.

Or we may refine $0$ as $0_1$ (oscillation in one direction at a high angular speed), $0_2$ (oscillation in the same direction at a lower angular speed), etc.

Or we may refine $1$ as $1_1$ (oscillation in opposite direction at a high angular speed), $1_2$ (oscillation in the same opposite direction at a lower angular speed), etc.

The refinement of the neutrosophic polarization may be given by one or more parameters that influence the oscillation of the photon.

## *XIII.6.4. Neutrosophic Quantum Computer.*

A *Neutrosophic Quantum Computer* uses phenomena of Neutrosophic Quantum Mechanics, such as neutrosophic superposition and neutrosophic entanglement for neutrosophic data operations.

## *XIII.6.5. Neutrosophic Particle.*

A *particle* is considered *neutrosophic* if it has some indeterminacy with respect to at least one of its attributes (direction of spinning, speed, charge, etc.).

## *XIII.6.6. Entangled Neutrosophic Particle.*

Two *neutrosophic particles* are *entangled* if measuring the indeterminacy of one of them, the other one will automatically have the same indeterminacy.

## *XIII.6.7. Neutrosophic Data.*

*Neutrosophic Data* is data with some indeterminacy.

## *XIII.6.8. Neutrosophic Superposition.*

*Neutrosophic Superposition*, that we introduce now for the first time, means superpositions only of *0* and *1* as in qubit (=quantum bit), but also involving indeterminacy (*I*), as in neutrosophic set, neutrosophic logic, neutrosophic probability, neutrosophic measure, and so on.

## *XIII.6.9. Indeterminate Bit.*

An *indeterminate bit*, that we introduce now for the first time, is a bit that one does not know if it is *0* or *1*, so we note it by *I* (= indeterminacy).

Therefore, neutrosophic superposition means coherent superposition of *0* and *I*, *1* and *I*, or *0* and *1* and *I*:

$$\begin{pmatrix} 0 \\ I \end{pmatrix}, \begin{pmatrix} 1 \\ I \end{pmatrix}, \text{ or } \begin{pmatrix} 0 \\ 1 \\ I \end{pmatrix},$$

or decoherent superposition of classical bits *0* and *1*, or decoherence between *0, 1, I,* such as:

$$\begin{pmatrix} 0 \\ 1 \end{pmatrix}_{dec}, \begin{pmatrix} 0 \\ I \end{pmatrix}_{dec}, \begin{pmatrix} 1 \\ I \end{pmatrix}_{dec}, \begin{pmatrix} 0 \\ 1 \\ I \end{pmatrix}_{dec}.$$

## *XIII.6.10. Neutrobit.*

A *neutrosophic bit* (or "neutrobit"), that we also introduce for the first time, is any of the above neutrosophic superpositions:

$$\begin{pmatrix} 0 \\ I \end{pmatrix}, \begin{pmatrix} 1 \\ I \end{pmatrix}, \begin{pmatrix} 0 \\ 1 \\ I \end{pmatrix}, \text{ or } \begin{pmatrix} 0 \\ 1 \end{pmatrix}_{dec}.$$

A neutrosobit acts in two or three universes. A neutrobit can exist with, of course, a *(t, i, f)*

neutrosophic probability, simultaneously as *0* and *I*, or *1* and *I*, or *0, 1,* and *I*, where *t* = percentage of truth, *i* = percentage of indeterminacy, and *f* = percentage of falsehood.

## *XIII.6.11. Refined Neutrosophic Quantum Computer.*

Thus, we extend the neutrosophic quantum computers to *refined neutrosophic quantum computers*.

## *XIII.6.12. Neutrosophic Filter Polarization.*

The *neutrosophic filter polarization* of the receiver must match the neutrosophic polarization of the transmitter, of course.

## *XIII.6.13. Neutrosophic Quantum Parallelism.*

The *neutrosophic quantum parallelism* is referring to the simultaneously calculations done in each universe, but some universe may contain indeterminate bits, or there might be some decoherence superpositions.

## *XIII.6.14. n-Neutrobit Quantum Computer.*

Thus, an *n-neutrobit quantum computer*, whose register has *n* neutrobits, requires $3^n - 1$ numbers created from the digits *0, 1,* and *I* (where *I* is considered as an indeterminate digit).

A register of *n* classical bits represents any number from *0* to $2^n - 1$. A register of *n* qubits such that each bit is in superposition or coherent state, can represent simultaneously all numbers from *0* to $2^n - 1$.

Being in neutrosophic superposition, a neutrosophic quantum computer can simultaneously act on all its possible states.

## *XIII.6.15. Neutrosophic Quantum Gates.*

Moving towards *neutrosophic quantum gates* involves experiments in which one observes quantum phenomena with indeterminacy.

## *XIII.6.16. Remarks.*

Building a *Neutrosophic Quantum Computer* requires a neutrosophic technology that enables the “neutrobits”, either with coherent super-

positions involving *I*, or with decoherent superpositions.

Since neither classical quantum computers have been built yet, neutrosophic quantum computers would be as today even more difficult to construct.

But we are optimistic that they will gather momentum in practice one time in the future.

## *XIII.6.17. Reversibility of a Neutrosophic Quantum Computer.*

The *reversibility of a neutrosophic quantum computer* is more problematic than that of a classical quantum computer, since amongst its neutrosophic inputs that must be entirely deducible from its neutrosophic outputs, there exists *I* (indeterminacy).

This becomes even more complex when one deals with refined neutrosophic polarisations, such as sub-indeterminacies $(I_1, I_2)$ and sub-oscillations in one direction, or in another direction.

A loss of neutrosophic information (i.e. information with indeterminacy) results from

*irreversible neutrosophic quantum computers* (when its inputs are not entirely deducible from its outputs). The loss of information, which comes from the loss of heat of the photons, means loss of bits, or qubits, or neutrobits.

## *XIII.6.18. Neutrosophic Dynamical System.*

Any classical dynamical system is, in some degree neutrosophic, since any dynamical system has some indeterminacy because a dynamic system is interconnected with its environment, hence interconnected with other dynamical systems.

We can, in general, take any *neutrosophic dynamical system*, as a neutrosophic quantum computer, and its dynamicity as a *neutrosophic computation*.

## *XIII.6.19. Neutrosophic Turing Machine & Neutrosophic Church-Turing Principle.*

We may talk about a *Neutrosophic Turing Machine*, which is a Turing Machine which works approximately (hence it has some indeterminacy), and about a *Neutrosophic Church-Turing Principle*,

which deviates and extends the classical Church-Turing Principle to:

"There exists or can be built a universal 'neutrosophic quantum' [NB: *our inserted words*] that can be programmed to perform any computational task that can be performed by any physical object."

## *XIII.6.20. Human Brain as an example of Neutrosophic Quantum Computer.*

As a particular case, the human brain is a neutrosophic quantum computer (the neutrosophic hardware), since it works with indeterminacy, vagueness, unknown, incomplete and conflicting information from our-world. And because it processes simultaneously information in conscience and sub-conscience (hence netrosophic parallelism). The human mind is neutrosophic software, since works with approximations and indeterminacy.

## *XIII.6.21. Neutrosophic Quantum Dot.*

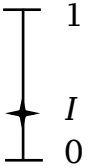
In the classical theoretical quantum computers, a *quantum dot* is represented by one electron

contained into a cage of atoms. The electron at the ground state is considered the *0* state of the classical qubit, while the electron at the excited (that is caused by a laser light pulse of a precise duration and wavelength) is considered the *1* state of the classical qubit.

When the laser light pulse that excites the electron is only half of the precise duration, the electron gets in a classical superposition of *0* and *1* states simultaneously.

A right duration-and-wavelength laser light pulse knocks the electron from *0* to *1*, or from *1* to *0*. But, when the laser light pulse is only a fraction of the right duration, then the electron is placed in between the ground state (*0*) and the excited state (*1*), i.e. the electron is placed in indeterminate state (*I*). We denote the indeterminate state by "*I*", as in neeutrosophic logic, and of course $I \in (0,1)$ in this case.

Hence, one has a *refined neutrosophic logic*, where the indeterminacy is refined infinitely many times, whose values are in the open interval *(0, 1).* Such as




This is a neutrosophication process.

## *XIII.6.22. Neutrosophic NOT Function.*

The *controlled neutrosophic NOT function* is defined by the laser-light application:

$NOT_N: [0,1] \to [0,1]$.

$NOT_N(x) = 1 - x$, where $x \in [0,1]$.

Therefore:

$NOT_N(0) = 1$, $NOT_N(1) = 0$,

and

$NOT_N(I) = 1 - I$.

For example, if indeterminacy $I = 0.3$, then

$NOT_N(0.3) = 1 - 0.3 = 0.7$.

Hence $NOT_N$ (indeterminacy) = indeterminacy.

## *XIII.6.23. Neutrosophic AND Function.*

The neutrosophic AND function is defined as:

$AND_N: [0,1] \times [0,1] \to [0,1]$.

$AND_N(x,y) = min\{x,y\}$, for all $x,y \in [0,1]$.

Therefore:

$AND_N(0,0) = 0$, $AND_N(1,1) = 1$,

$AND_N(0,1) = AND_N(1,0) = 0.$

For indeterminacy,

$AND_N(0,I) = 0$, and $AND_N(1,I) = I$.

Let $I = 0.4$, then:

$AND_N(0,0.4) = 0$, $AND_N(1,0.4) = 0.4$.

Another example with indeterminacies.

$AND_N(0.4,0.6) = 0.4$.

## *XIII.6.24. Neutrosophic OR Function.*

The neutrosophic OR function is defined as:

$OR_N: [0,1] \times [0,1] \to [0,1]$.

$OR_N(x,y) = max\{x,y\}$, for all $x,y \in [0,1]$.

Therefore:

$OR_N(0,0) = 0$, $OR_N(1,1) = 1$,

$OR_N(0,1) = O, R_N(1,0) = 0$.

For indeterminacy,

$OR_N(0,I) = I$, and $OR_N(1,I) = 1$.

If $I = 0.2$, then $OR_N(0,0.2) = 0.2$, and $OR_N(1,0.2) = 0.2$.

## *XIII.6.25. Neutrosophic IFTHEN Function.*

The neutrosophic $IFTHEN_N$ function is defined as:

$IFTHEN_N: [0,1] \times [0,1] \to [0,1]$.

$IFTHEN_N(x,y) = max\{1 - x, y\}$, for all $x, y \in [0,1]$.

$IFTHEN_N$ is equivalent to $OR_N(NOT_N(x), y)$, similar to the Boolean logic:

$A \to B$ is equivalent to $non(A)$ or $B$.

Therefore:

$IFTHEN_N(0,0) = 1$, $IFTHEN_N(1,1) = 1$,

$IFTHEN_N(1,0) = 0$, $IFTHEN_N(0,1) = 1$.

Its neutrosophic value table is:

$IFTHEN_N$

| x / y | 0 | $I_\alpha$ | $I_\beta$ | 1 |
|---|---|---|---|---|
| 0 | 1 | $1- I_\alpha$ | $1- I_\beta$ | 0 |
| $I_\alpha$ | 1 | $\max\{1- I_\alpha, I_\alpha\}$ | $\max\{1- I_\beta, I_\alpha\}$ | $I_\alpha$ |
| $I_\beta$ | 1 | $\max\{1- I_\alpha, I_\beta\}$ | $\max\{1- I_\beta, I_\beta\}$ | $I_\beta$ |
| 1 | 1 | 1 | 1 | 1 |

where $I_\alpha$, $I_\beta$ are indeterminacies and they belong to $(0,1)$.

$I_\alpha$, $I_\beta$ can be crisp numbers, interval-valued, or in general subsets of [0, 1].

## *XIII.6.16. Neutrosophic Quantum Liquids.*

In classical theoretical quantum computers, there also are used *computing liquids*. In order to store the information, one employs a soup of complex molecules, i.e. molecules with many nuclei. If a molecule is sunk into a magnetic field, each of its nuclei spins either downward (which means state *0*), or upward (which means state *1*).

Precise radio waves bursts change the nuclei spinning from *0* to *1*, and reciprocally. If the radio waves are not at a right amplitude, length and frequency, then the nuclei state is perturbed (which means neither *0* nor *1*, but *I* = indeterminacy). Similarly, this is a *neutrosophication process*.

These spin states (*0, 1,* or *I*) can be detected with the techniques of NNMR (*Neutrosophic Nuclear Magnetic Resonance*).

The *deneutrosophication* means getting rid of indeterminacy (noise), or at least diminish it as much as possible.

## *XIII.6.27. Conclusion.*

This is a theoretical approach and investigation about the possibility of building a quantum computer based on neutrosophic logic. Future research in this direction is required.

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# *XIII.7. Theory of Neutrosophic Evolution: Degrees of Evolution, Indeterminacy, and Involution*

## *Abstract*

During the process of adaptation of a being (plant, animal, or human), to a new environment or conditions, the being partially *evolves*, partially *devolves* (degenerates), and partially is *indeterminate* {i.e. neither evolving nor devolving, therefore unchanged (neutral), or the change is unclear, ambiguous, vague}, as in neutrosophic logic. Thank to adaptation, one therefore has: *evolution*, *involution*, and i*ndeterminacy* (or *neutrality*), each one of these three neutrosophic components in some degree.

The degrees of evolution / indeterminacy / involution are referred to both: the *structure* of the being (its body parts), and *functionality* of the being (functionality of each part, or inter-functionality of the parts among each other, or functionality of the being as a whole).

We therefore introduce now for the first time the *Neutrosophic Theory of Evolution, Involution, and Indeterminacy (or Neutrality).*

## *XIII.7.1. Introduction.*

During the 2016-2017 winter, in December-January, I went to a cultural and scientific trip to Galápagos Archipelago, Ecuador, in the Pacific Ocean, and visited seven islands and islets: Mosquera, Isabela, Fernandina, Santiago, Sombrero Chino, Santa Cruz, and Rabida, in a cruise with Golondrina Ship. I had extensive discussions with our likeable guide, señor Milton Ulloa, about natural habitats and their transformations.

After seeing many animals and plants, that evolved differently from their ancestors that came from the continental land, I consulted, returning back to my University of New Mexico, various scientific literature about the life of animals and plants, their reproductions, and about multiple theories of evolutions. I used the online scientific databases that UNM Library [25] has subscribed to, such as: MathSciNet, Web of Science, EBSCO, Thomson Gale (Cengage), ProQuest, IEEE/IET

Electronic Library, IEEE Xplore Digital Library etc., and DOAJ, Amazon Kindle, Google Play Books as well, doing searches for keywords related to origins of life, species, evolution, controversial ideas about evolution, adaptation and inadaptation, life curiosities, mutations, genetics, embryology, and so on.

My general conclusion was that each evolution theory had some degree of truth, some degree of indeterminacy, and some degree of untruth (as in neutrosophic logic), depending on the types of species, environment, timespan, and other hidden parameters that may exist.

And all these degrees are different from a species to another species, from an environment to another environment, from a timespan to another timespan, and in general from a parameter to another parameter.

By environment, one understands: geography, climate, prays and predators of that species, i.e. the whole ecosystem.

I have observed that the animals and plants (and even human beings) not only *evolve*, but also

*devolve* (i.e. involve back, decline, atrophy, pass down, regress, degenerate). Some treats increase, other treats decrease, while others remains unchanged (neutrality).

One also sees: adaptation by physical or functional evolution of a body part, and physical or functional involution of another body part, while other body parts and functions remain unchanged. After evolution, a new process starts, re-evaluation, and so on.

In the society, it looks that the most opportunistic (which is the fittest!) succeeds, not the smartest. And professional deformation signifies evolution (specialization in a narrow field), and involution (incapability of doing things in another field).

The paper is organized as follows: some information on taxonomy, species, a short list of theories of origin of life, another list of theories and ideas about evolution. Afterwards the main contribution of this paper, the *theory of neutrosophic evolution*, the dynamicity of species, several examples of evolution, involution, and

indeterminacy (neutrality), neutrosophic selection, refined neutrosophic theory of evolution, and the paper ends with open questions on evolution / neutrality / involution.

## *XIII.7.2. Taxonomy.*

Let's recall several notions from classical biology.

The *taxonomy* is a classification, from a scientifically point of view, of the living things, and it classifies them into three categories: *species, genus,* and *family*.

## *XIII.7.3. Species.*

A *species* means a group of organisms, living in a specific area, sharing many characteristics, and able to reproduce with each other.

In some cases, the distinction between a population subgroup to be a different species, or not, is unclear, as in the *Sorites Paradoxes* in the frame of *neutrosophy*: the frontier between <A> (where <A> can be a species, a genus, or a family), and <nonA> (which means that is not <A>) is

vague, incomplete, ambiguous. Similarly, for the distinction between a series and its subseries.

## *XIII.7.4. Theories of Origin of Life.*

Louis Pasteur (1822-1895) developed in 1860 the theory of *precellular (prebiotic) evolution*, which says that life evolved from non-living chemical combinations that, over long time, arose spontaneously.

In the late 19th century a theory, called *abiogenesis,* promulgated that the living organisms originated from lifeless matter spontaneously, without any living parents' action.

Carl R. Woese (b. 1928) has proposed in 1970's that the *progenotes* were the very first living cells, but their biological specificity was small. The genes were considered probable (rather than identical) proteins.

John Burdon Sanderson Haldane (1872-1964) proposed in 1929 the *theory that the viruses were precursors to the living cells* [1].

John Bernal and A. G. Cairns-Smith stated in 1966 *the mineral theory:* that life evolved from

inorganic crystals found in the clay, by natural selection [2].

According to the *little bags theory of evolution*, the life is considered as having evolved from organic chemicals that happened to get trapped in some tiny vesicles.

Eigen and Schuster, adepts of *the hypercycle theory*, asserted in 1977 that the precursors of single cells were these little bags, and their chemical reactions cycles were equivalent to the life's functionality [3].

Other theories about the origin of life have been proposed in the biology literature, such as: *primordial soup, dynamic state theory,* and *phenotype theory*, but they were later dismissed by experiments.

## *XIII.7.5. Theories and Ideas about Evolution.*

The theory of *fixism* says that species are fixed, they do not evolve or devolve, and therefore the today's species are identical to the past species.

Of course, the *creationism* is a fixism theory, from a religious point of view. Opposed to the

fixism is the theory of *transformism*, antecedent to the evolutionary doctrine, in the pre-Darwinian period, which asserts that plants and animals are modified and transformed gradually from one species into another through many generations [22].

Jean Baptiste Pierre Antoine de Monet Lamarck (1749-1829), in 1801, ahead of Charles Darwin, is associated with the *theory of inheritance of acquired characteristics* (or *use-inheritance*), and even *of acquired habits*. Which is called *Lamarckism* or *Lamarckian Evolution*.

If an animal repeatedly stresses in the environment, its body part under stress will modify in order to overcome the environmental stress, and the modification will be transmitted to its offspring.

For example: the giraffe having a long neck in order to catch the tree leaves [4].

Herbert Spencer (1820-1903) used for the first time the term *evolution* in biology, showing that a population's gene pool changes from a generation

to another generation, producing new species after a time [5].

Charles Darwin (1809-1882) introduced the *natural selection*, meaning that individuals that are more endowed with characteristics for reproduction and survival will prevail ("selection of the fittest"), while those less endowed would perish [6].

Darwin had also explained the structure similarities of leaving things in genera and families, due to the common descent of related species [7].

In his *gradualism* (or *phyletic gradualism*), Darwin said that species evolve slowly, rather than suddenly.

The adaptation of an organism means nervous response change, after being exposed to a permanent stimulus.

In the *modern gradualism*, from the genetic point of view, the beneficial genes of the individuals best adapted to the environment, will have a higher frequency into the population over a period of time, giving birth to a new species [8].

Herbert Spencer also coined the phrase *survival of the fittest* in 1864, that those individuals the best adapted to the environment are the most likely to survive and reproduce.

Alfred Russel Wallace (1823-1913) coined in 1828 the terms *Darwinism* (individuals the most adapted to environment pass their characteristics to their offspring), and *Darwinian fitness* (the better adapted, the better surviving chance) [9].

One has upward evolution {*anagenesis*, coined by Alpheus Hyatt (1838-1902) in 1889}, as the progressive evolution of the species into another [10], and a *branching evolution* {*cladogenesis*, coined by Sir Julian Sorell Huxley (1887-1975) in 1953}, when the population diverges and new species evolve [11].

George John Romanes (1848-1894) coined the word *neo-Darwinism*, related to natural selection and the theory of genetics that explains the *synthetic theory of evolution*. What counts for the natural selection is the gene frequency in the population [12]. The Darwinism is put together

with the paleontology, systematics, embryology, molecular biology, and genetics.

In the 19th century Gregor Johann Mendel (1822-1884) set the base of *genetics*, together with other scientists, among them Thomas Hunt Morgan (1866-1945).

The *Mendelism* is the study of heredity according to the chromosome theory: the living thing reproductive cells contain factors which transmit to their offspring particular characteristics [13].

August Weismann (1834-1914) in year 1892 enounced the *germ plasm theory*, saying that the offspring do not inherit the acquired characteristics of the parents [14].

Hugo de Vries (1848-1935) published a book in 1901/1903 on *mutation theory*, considering that randomly genetic mutations may produce new forms of living things. Therefore, new species may occur suddenly [15].

Louis Antoine Marie Joseph Dollo (1857-1931) enunciated the *Dollo's principle* (*law* or *rule*) that evolution is irreversible, i.e. the lost functions and

structures in species are not regained by future evolving species.

In the present, the *synergetic theory of evolution* considers that one has a natural or artificial multipolar selection, which occurs at all life levels, from the molecule to the ecosystem – not only at the population level.

But nowadays it has been discovered organisms that have re-evolved structured similar to those lost by their ancestors [16].

The genetic assimilation {for *Baldwin Effect*, after James Mark Baldwin (1861-1934)} considered that an advantageous trait (or phenotype) may appear in several individuals of a population in response to the environmental cues, which would determine the gene responsible for the trait to spread through this population [17].

The British geneticist Sir Ronald A. Fisher (1890-1962) elaborated in 1930 *the evolutionary or directional determinism*, when a trait of individuals is preferred for the new generations (for example the largest grains to replant, chosen by farmers) [18].

The *theory of speciation* was associated with Ernst Mayr (b. 1904) and asserts that because of geographic isolation new species arise, that diverge genetically from the larger original population of sexually reproducing organisms. A subgroup becomes new species if its distinct characteristics allow it to survive and its genes do not mix with other species [19].

In the 20th century, Trofim Denisovitch Lysenko (1898-1976) revived the Lamarckism to the *Lysenkoism* school of genetics, proclaiming that the new characteristics acquired by parents will be passed on to the offspring [20].

Richard Goldschmidt (1878-1958) in 1940 has coined the terms of *macroevolution*, which means evolution from a long timespan (geological) perspective, and *microevolution*, which means evolution from a small timespan (a few generations) perspective with observable changes [1].

Sewall Wright (1889-1988), in the mid 20th century, developed the *founders effect of principle*, that in isolated places population

arrived from the continent or from another island, becomes little by little distinct from its original place population. This is explained because the founders are few in number and therefore the genetic pool is smaller in diversity, whence their offspring are more similar in comparison to the offspring of the original place population.

The founders effect or principle is regarded as a particular case of the *genetic drift* (by the same biologist, Sewall Wright), which tells that the change in gene occurs by chance [21].

The mathematician John Maynard Smith has applied the game theory to animal behavior and in 1976 he stated *the evolutionary stable strategy* in a population. It means that, unless the environment changes, the best strategy will evolve, and persist for solving problems.

Other theories related to evolution such as: *punctuated equilibrium* (instantaneous evolution), *hopeful monsters,* and *saltation (quantum) speciation* (that new species suddenly occur; by Ernst Mayr) have been criticized by the majority of biologists.

## *XIII.7.6. Open Research.*

By genetic engineering it is possible to make another combination of genes, within the same number of chromosomes. Thus, it is possible to mating a species with another closer species, but their offspring is sterile (the offspring cannot reproduce).

Despite the tremendous genetic engineering development in the last decades, there has not been possible to prove by experiments in the laboratory that: from an inorganic matter, one can make organic matter that may reproduce and assimilate energy; nor was possible in the laboratory to transform a species into a new species that has a number of chromosomes different from the existent species.

## *XIII.7.7. Involution.*

According to several online dictionaries, *involution* means:

— Decay, retrogression or shrinkage in size; or return to a former state [Collins Dictionary of Medicine, Robert M. Youngson, 2005];

— Returning of an enlarged organ to normal size; or turning inward of the edges of a part; mental decline associated with advanced age (psychiatry) [Medical Dictionary for the Health Professions and Nursing, Farlex, 2012];

— Having rolled-up margins (for the plant organs) [Collins Dictionary of Biology, 3rd edition, W. G. Hale, V. A. Saunders, J. P. Margham, 2005];

— A retrograde change of the body or of an organ [Dorland's Medical Dictionary for Health Consumers, Saunders, an imprint of Elsevier, Inc., 2007];

— A progressive decline or degeneration of normal physiological functioning [The American Heritage, Houghton Mifflin Company, 2007].

## *XIII.7.8. Theory of Neutrosophic Evolution.*

During the process of adaptation of a being (plant, animal, or human) B, to a new environment η,

— B partially *evolves*;

— B partially *devolves* (involves, regresses, degenerates);

— and B partially remains *indeterminate* {which means *neutral* (unchanged), or ambiguous – i.e. not sure if it is evolution or involution}.

Any action has a reaction. We see, thank to adaptation: evolution, involution, and neutrality (indeterminacy), each one of these three *neutrosophic components* in some degree.

The degrees of evolution / indeterminacy / involution are referred to both: the *structure* of B (its body parts), and *functionality* of B (functionality of each part, or inter-functionality of the parts among each other, or functionality of B as a whole).

*Adaptation* to new environment conditions means *de-adaptation* from the old environment conditions.

Evolution in one direction means involution in the opposite direction.

Loosing in one direction, one has to gain in another direction in order to survive (for equilibrium). And reciprocally.

A species, with respect to an environment, can be:

— in equilibrium, disequilibrium, or indetermination;

— stable, unstable, or indeterminate (ambiguous state);

— optimal, suboptimal, or indeterminate.

One therefore has a *Neutrosophic Theory of Evolution, Involution, and Indeterminacy* (neutrality, or fluctuation between Evolution and Involution). The evolution, the involution, and the indeterminate-evolution depend not only on natural selection, but also on many other factors such as: artificial selection, friends and enemies, bad luck or good luck, weather change, environment juncture etc.

## *XIII.7.9. Dynamicity of the Species.*

If the species is in indeterminate (unclear, vague, ambiguous) state with respect to its environment, it tends to converge towards one extreme:

either to equilibrium / stability / optimality, or to disequilibrium / instability / suboptimality with respect to an environment;

therefore, the species either rises up gradually or suddenly by mutation towards equilibrium / stability / optimality;

or the species deeps down gradually or suddenly by mutation to disequilibrium / instability / suboptimality and perish.

The *attraction point* in this neutrosophic dynamic system is, of course, the state of equilibrium / stability / optimality. But even in this state, the species is not fixed, it may get, due to new conditions or accidents, to a degree of disequilibrium / instability / suboptimality, and from this new state again the struggle on the long way back of the species to its attraction point.

## *XIII.7.10. Several Examples of Evolution, Involution, and Indeterminacy (Neutrality)*

### *XIII.7.10.1 Cormorants Example*

Let's take the flightless cormorants (*Nannopterum harrisi*) in Galápagos Islands, their wings and tail have atrophied (hence *devolved*) due to their no need to fly (for they are having no predators on the land), and because their

permanent need to dive on near-shore bottom after fish, octopi, eels etc. Their avian breastbone vanished (*involution*), since no flying muscles to support were needed.

But their neck got longer, their legs stronger, and their feet got huge webbed in order to catch fish underwater (*evolution*).

Yet, the flightless cormorants kept several of their ancestors' habits (functionality as a whole): make nests, hatch the eggs etc. (hence *neutrality*).

### *XIII.7.10.2. Cosmos Example.*

The astronauts, in space, for extended period of time get accustomed to low or no gravity (*evolution*), but they lose bone density (*involution*). Yet other body parts do not change, or it has not been find out so far (*neutrality / indeterminacy*).

### *XIII.7.10.3. Example of Evolution and Involution*

The whales *evolved* with respect to their teeth from pig-like teeth to cusped teeth. Afterwards, the whales *devolved* from cusped teeth back to conical teeth without cusps.

### *XIII.7.10.4. Penguin Example.*

The Galápagos Penguin (*Spheniscus mendiculus*) *evolved* from the Humboldt Penguin by shrinking its size at 35 cm high (adaptation by *involution*) in order to be able to stay cool in the equatorial sun.

### *XIII.7.10.5. Frigate Birds Example.*

The Galápagos Frigate birds are birds that lost their ability to dive for food, since their feathers are not waterproof (*involution*), but they became masters of faster-and-maneuverable flying by stealing food from other birds, called klepto-parasite feeding (*evolution*).

### *XIII.7.10.6. Example of Darwin's Finches.*

The 13 Galápagos species of Darwin's Finches manifest various degrees of evolution upon their beak, having different shapes and sizes for each species in order to gobble different types of foods (hence *evolution*):

— for cracking hard seeds, a thick beak (ground finch);

— for insects, flowers and cacti, a long and slim beak (another finch species).

Besides their beaks, the finches look similar, proving they came from a common ancestor (hence *neutrality*).

If one experiments, let's suppose one moves the thick-beak ground finches back to an environment with soft seeds, where it is not needed a thick beak, then the thick beak will atrophy and, in time, since it becomes hard for the finches to use the heavy beak, the thin-beak finches will prevail (hence *involution*).

*XIII.7.10.7. El Niño Example.*

Professor of ecology, ethology, and evolution Martin Wikelski, from the University of Illinois at Urbana – Champaign, has published in the journal "Nature" a curious report, regarding data he and his team collected about marine iguanas since 1987.

During the 1997 – 1998 El Niño, the marine algae died, and because the lack of food, on one of the Galápagos islands some marine iguanas shrank a quarter of their length and lost half of their weight (adaptation by *involution*).

After plentiful of food became available again, the marine iguanas grew back to their original length and weight (re-adaptation by *evolution*). [26]

*XIII.7.10.8. Bat Example.*

The bats are the only mammals capable of naturally flying, due to the fact that their fore-limbs have developed into webbed wings (*evolution* by transformation). But navigating and foraging in the darkness, have caused their eyes' functionality to diminish (*involution*), yet the bats "see" with their ears (*evolution* by transformation) using the echolocation (or the bio sonar) in the following way: the bats emit sounds by mouth (one emitter), and their ears receive echoes (two receivers); the time delay (between emission and reception of the sound) and the relative intensity of the received sound give to the bats information about the distance, direction, size and type of animal in its environment.

*XIII.7.10.9. Mole Example.*

For the moles, mammals that live underground, their eyes and ears have degenerated and become

minuscule since their functions are not much needed (hence adaptation by *involution*), yet their forelimbs became more powerful and their paws larger for better digging (adaptation by *evolution*).

## *XIII.7.11. Neutrosophic Selection*

Neutrosophic Selection with respect to a population of a species means that over a specific timespan a percentage of its individuals evolve, another percentage of individuals devolve, and a third category of individuals do not change or their change is indeterminate (not knowing if it is evolution or involution). We may have a natural or artificial neutrosophic selection.

## *XIII.7.12. Refined Neutrosophic Theory of Evolution*

Refined Neutrosophic Theory of Evolution is an extension of the neutrosophic theory of evolution, when the degrees of evolution / indeterminacy / involution are considered separately with respect to each body part, and with respect to each body part functionality, and with respect to the whole organism functionality.

## *XIII.7.13. Open Questions on Evolution / Neutrality / Involution.*

XIII.7.13.1. How to measure the degree of evolution, degree of involution, and degree of indeterminacy (neutrality) of a species in a given environment and a specific timespan?

XIII.7.13.2. How to compute the degree of similarity to ancestors, degree of dissimilarity to ancestors, and degree of indeterminate similarity-dissimilarity to ancestors?

XIII.7.13.3. Experimental Question. Let's suppose that a partial population of species $S_1$ moves from environment $\eta_1$ to a different environment $\eta_2$; after a while, a new species $S_2$ emerges by adaptation to $\eta_2$; then a partial population $S_2$ moves back from $\eta_2$ to $\eta_1$; will $S_2$ evolve back (actually devolve to $S_1$)?

XIII.7.13.4. Are all species needed by nature, or they arrived by accident?

## *XIII.7.14. Conclusion.*

We have introduced for the first time the concept of Neutrosophic Theory of Evolution, Indeterminacy (or Neutrality), and Involution.

For each being, during a long timespan, there is a process of partial evolution, partial indeterminacy or neutrality, and partial involution with respect to the being body parts and functionalities.

The function creates the organ. The lack of organ functioning, brings atrophy to the organ.

In order to survive, the being has to adapt. One has adaptation by evolution, or adaptation by involution – as many examples have been provided in this paper. The being partially evolves, partially devolves, and partially remains unchanged (fixed) or its process of evolution–involution is indeterminate. There are species partially adapted and partially struggling to adapt.

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## *XIII.8. Neutrosophic Triplet Structures in Practice*

This new field of neutrosophic triplet structures is important, because it reflects our everyday life [it is not simple imagination!].

The neutrosophic triplets are based on real triads: (friend, neutral, enemy), (positive particle, neutral particle, negative particle), (yes, undecided, no), (pro, neutral, against), and in general *(<A>, <neutA>, <antiA>)* as in neutrosophy.

These neutrosophic triplet structures will be more practical than the classical algebraic structures – because the last ones are getting more and more abstract and too idealistic.

This book is part of the book-series dedicated to the advances of neutrosophic theories and their applications, started by the author in 1998. Its aim is to present the last developments in the field.

This is the second extended and improved edition of *Neutrosophic Perspectives* (September 2017; first edition was published in June 2017).

For the first time, we now introduce:

— Neutrosophic Duplets and the Neutrosophic Duplet Structures;

— Neutrosophic Multisets (as an extension of the classical multisets);

— Neutrosophic Spherical Numbers;

— Neutrosophic Overnumbers / Undernumbers / Offnumbers;

— Neutrosophic Indeterminacy of Second Type;

— Neutrosophic Hybrid Operators (where the heterogeneous t-norms and t-conorms may be used in designing neutrosophic aggregations);

— Neutrosophic Triplet Loop;

— Neutrosophic Triplet Function;

— Neutrosophic Modal Logic;

— and Neutrosophic Hedge Algebras.

The Refined Neutrosophic Set / Logic / Probability were introduced in 2013 by F. Smarandache. Since year 2016 a new interest has been manifested by researchers for the Neutrosophic Triplets and their corresponding Neutrosophic Triplet Algebraic Structures (introduced by F. Smarandache & M. Ali). Subtraction and Division of Neutrosophic Numbers were introduced by F. Smarandache - 2016, and Jun Ye – 2017.

We also present various new applications in: neutrosophic multi-criteria decision-making, neutrosophic psychology, neutrosophic geographical function (the equatorial virtual line), neutrosophic probability in target identification, neutrosophic dynamic systems, neutrosophic quantum computers, neutrosophic theory of evolution, and neutrosophic triplet structures in our everyday life.

*The Author.*

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